Pluriassociative algebras II: The polydendriform operad and related operads

Samuele Giraudo

PLURIASSOCIATIVE ALGEBRAS II: THE POLYDENDRIFORM OPERAD AND RELATED OPERADS

SAMUELE GIRAUDO

ABSTRACT. Dendriform algebras form a category of algebras recently introduced by Loday. A dendriform algebra is a vector space endowed with two nonassociative binary operations satisfying some relations. Any dendriform algebra is an algebra over the dendriform operad, the Koszul dual of the diassociative operad. We introduce here, by adopting the point of view and the tools offered by the theory of operads, a generalization on a nonnegative integer parameter γ of dendriform algebras, called γ-polydendriform algebras, so that 1-polydendriform algebras are dendriform algebras. For that, we consider the operads obtained as the Koszul duals of the γ-pluriassociative operads introduced by the author in a previous work. In the same manner as dendriform algebras are suitable devices to split associative operations into two parts, γ-polydendriform algebras seem adapted structures to split associative operations into 2γ operation so that some partial sums of these operations are associative. We provide a complete study of the γ-polydendriform operads, the underlying operads of the category of γ-polydendriform algebras. We exhibit several presentations by generators and relations, compute their Hilbert series, and construct free objects in the corresponding categories. We also provide consistent generalizations on a nonnegative integer parameter of the duplicial, triassociative and tridendriform operads, and of some operads of the operadic butterfly.

CONTENTS

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Introduction

Associative algebras play an obvious and primary role in algebraic combinatorics. In recent years, the study of natural operations on certain sets of combinatorial objects has given rise to more or less complicated algebraic structures on the vector spaces spanned by these sets. A primordial point to observe is that these structures maintain furthermore many links with combinatorics, combinatorial Hopf algebra theory, representation theory, and theoretical physics. Let us cite for instance the algebra of symmetric functions [Mac95] involving integer partitions, the algebra of noncommutative symmetric functions [GKL+95] involving integer compositions, the Malvenuto-Reutenauer algebra of free quasi-symmetric functions [MR95] (see also [DHT02]) involving permutations, the Loday-Ronco Hopf algebra of binary trees [LR98] (see also [HNT05]), and the Connes-Kreimer Hopf algebra of forests of rooted trees [CK98].

There are several ways to understand and to gather information about such structures. A very fruitful strategy consists in splitting their associative products $\star$ into two separate operations $\prec$ and $\succ$ in such a way that $\star$ turns to be the sum of $\prec$ and $\succ$. To be more precise, if $\mathcal{V}$ is a vector space endowed with an associative product $\star$, splitting $\star$ consists in providing two operations $\prec$ and $\succ$ defined on $\mathcal{V}$ and such that for all elements x and y of $\mathcal{V}$,

$$x \star y = x \prec y + x \succ y. \qquad (0.0.1)$$

This splitting property is more concisely denoted by

$$\star = \prec + \succ . \qquad (0.0.2)$$

One of the most obvious example occurs by considering the shuffle product on words. Indeed, this product can be separated into two operations according to the origin (first or second operand) of the last letter of the words appearing in the result [Ree58]. Other main examples include the split of the shifted shuffle product of permutations of the Malvenuto-Reutenauer Hopf algebra and of the product of binary trees of the Loday-Ronco Hopf algebra [Foi07]. The original formalization and the germs of generalization of these notions, due to Loday [Lod01], lead to the introduction of dendriform algebras. Dendriform algebras are vector spaces endowed with two operations $\prec$ and $\succ$ so that $\prec + \succ$ is associative and satisfy few other relations. Since any dendriform algebra is a quotient of a certain free dendriform algebra, the study of free dendriform algebras is worthwhile. Besides, the description of free dendriform algebras has a nice combinatorial interpretation involving binary trees and shuffle of binary trees.

In recent years, several generalizations of dendriform algebras were introduced and studied. Among these, one can cite dendriform trialgebras [LR04], quadri-algebras [AL04], ennea-algebras [Ler04], m-dendriform algebras of Leroux [Ler07], and m-dendriform algebras of Novelli [Nov14], all providing new ways to split associative products into more than two pieces. Besides, free objects in the corresponding categories of these algebras can be described by relatively complex combinatorial objects and more or less tricky operations on these. For instance, free dendriform trialgebras involve Schröder trees, free quadri-algebras involve noncrossing connected graphs on a circle, and free m-dendriform algebras of Leroux and free m-dendriform algebras of Novelli involves planar rooted trees where internal nodes have a constant number of children.

The theory of operads (see [LV12] for a complete exposition and also [Cha08]) seems to be one of the best tools to put all these algebraic structures under a same roof. Informally, an operad is a space of abstract operators that can be composed. The main interest of this theory is that any operad encodes a category of algebras and working with an operad amounts to work with the algebras all together of this category. Moreover, this theory gives a nice translation of connections that may exist between *a priori* two very different sorts of algebras. Indeed, any morphism between operads gives rise to a functor between the both encoded categories. We have to point out that operads were first introduced in the context of algebraic topology [May72, BV73] but they are more and more present in combinatorics [Cha08].

The first goal of this work is to define and justify a new generalization of dendriform algebras. Our long term primary objective is to develop new implements to split associative products in smaller pieces. Our main tool is the Koszul duality of operads, an important part of the theory introduced by Ginzburg and Kapranov [GK94]. We use the approach consisting in considering the diassociative operad Dias [Lod01], the Koszul dual of the dendriform operad Dendr, rather that focusing on Dendr. For this, we rely on the definition of a generalization Dias_γ on a nonnegative integer parameter γ of the diassociative operad introduced by the author in [Gir16]. These operads, called γ-pluriassociative operads, satisfy several properties and are among other set-operads and Koszul operads. We introduce in the present work the operads Dendr_γ as the Koszul dual of the operads Dias_γ.

The operads Dendr_γ are the underlying operads of the category of γ-polydendriform algebras, that are algebras with 2γ operations $\leftharpoonup_a$, $\rightharpoonup_a$, $a \in [\gamma]$, satisfying some relations. Free objects in these categories involve binary trees where all edges connecting two internal nodes are labeled on $[\gamma]$ and the computation of a product of two binary trees admits an inductive description. Moreover, the introduction of γ-polydendriform algebras offers to split an associative product $\star$ by

$$\star = \leftharpoonup_1 + \rightharpoonup_1 + \cdots + \leftharpoonup_\gamma + \rightharpoonup_\gamma, \tag{0.0.3}$$

with, among others, the stiffening conditions that all partial sums

$$\leftharpoonup_1 + \rightharpoonup_1 + \cdots + \leftharpoonup_a + \rightharpoonup_a \tag{0.0.4}$$

are associative for all $a \in \{1, \dots, \gamma\}$. Moreovoer, this work naturally leads to the consideration and the definition of numerous new operads. Table 1 summarizes some information about these.

This article is organized as follows. Section 1 contains the definition of the Koszul duality for operads and gives some recalls about the dendriform operad and dendriform algebras.

Then, the operad Dendr_γ is introduced in Section 2 as the Koszul dual of Dias_γ (Theorem 2.1.1). Since Dias_γ is a Koszul operad [Gir16], Dendr_γ also is, and then, by using results of Ginzburg and Kapranov [GK94], the alternating versions of the Hilbert series of Dias_γ and Dendr_γ are the inverses for each other for series composition. This, toghether with the expression for the Hilbert series of Dias_γ established in [Gir16], leads to an expression for the Hilbert series of Dendr_γ (Proposition 2.1.2). Motivated by the knowledge of the dimensions of Dendr_γ, we consider binary trees where internal edges are labelled on $\{1, \dots, \gamma\}$, called γ-edge valued binary trees. These trees form a generalization of the common binary trees indexing the bases

Operad	Objects	Dimensions	Symm.
Dendr_γ	γ-edge valued binary trees	$\gamma^{n-1}\frac{1}{n+1}\binom{2n}{n}$	No
As_γ	γ-corollas	γ	No
DAs_γ	γ-alternating Schröder trees	$\sum_{k=0}^{n-2}\gamma^{k+1}(\gamma-1)^{n-k-2}\frac{1}{k+1}\binom{n-2}{k}\binom{n-1}{k}$	No
Dup_γ	γ-edge valued binary trees	$\gamma^{n-1}\frac{1}{n+1}\binom{2n}{n}$	No
TDendr_γ	γ-edge valued Schröder trees	$\sum_{k=0}^{n-1}(\gamma+1)^k\gamma^{n-k-1}\frac{1}{k+1}\binom{n-1}{k}\binom{n}{k}$	No
Com_γ	—	—	Yes
Zin_γ	—	—	Yes

Table 1. The main operads defined in this paper. All these operads depend on a nonnegative integer parameter γ. The shown dimensions are the ones of the homogeneous components of arities $n \geqslant 2$ of the operads.

of Dendr, and index the bases of Dendr_γ. We continue the study of this operad by providing a new presentation obtained by considering the Koszul dual of Dias_γ over its K-basis, introduced in [Gir16] (Theorem 2.1.4). This presentation of Dendr_γ is very compact since its space of relations can be expressed only by three sorts of relations ((2.1.17a), (2.1.17b), and (2.1.17c)), each one involving two or three terms. We also describe all the associative elements of Dendr_γ over its two bases (Propositions 2.1.3, 2.1.5, and 2.1.6). We end this section by constructing the free γ-polydendriform algebra over one generator (Theorem 2.2.3). Its underlying vector space is the vector space of the γ-edge valued binary trees and is endowed with 2γ products described by induction. These products are kinds of shuffle of trees, generalizing the shuffle of trees introduced by Loday [Lod01] intervening in the construction of free dendriform algebras.

Section 3 extends a part of the operadic butterfly [Lod01, Lod06], a diagram of operads gathering the most classical ones together, including the diassociative, dendriform, and associative operads. To extends this diagram into our context, we introduce a generalization As_γ on a nonnegative integer parameter γ of the associative operad As. This operad, called γ-multiassociative operad, has γ associative generating operations, subjected to precise relations. We prove that this operad can be seen as a vector space of corollas labeled on $\{1, \dots, \gamma\}$ and that is Koszul (Proposition 3.1.1). Unlike the associative operad which is self-dual for Koszul duality, As_γ is not when $\gamma \geqslant 2$. The Koszul dual of As_γ, denoted by DAs_γ, is described by its presentation (Proposition 3.1.2) and is realized by means of γ-alternating Schröder trees, that are Schröder trees where internal nodes are labeled on $\{1, \dots, \gamma\}$ with an alternating condition (Proposition 3.1.5). In passing, we provide an alternative and simpler basis for the space of relations of DAs_γ than the one obtained directly by considering the Koszul dual of As_γ

(Proposition 3.1.3). We end this section by establishing a new version of the diagram gathering the diassociative, dendriform, and associative operads for the operads Dias_γ, As_γ, DAs_γ, and Dendr_γ (Theorem 3.2.3) by defining appropriate morphisms between these.

Finally, in Section 4, we sustain our previous ideas to propose generalizations on a nonnegative integer parameter γ of some more operads. We start by proposing a new operad Dup_γ generalizing the duplicial operad [Lod08], called γ-multiplicial operad. We prove that Dup_γ is Koszul and, like the bases of Dendr_γ, that the bases of Dup_γ are indexed by γ-edge valued binary trees (Proposition 4.1.2). The operads Dendr_γ and Dup_γ are nevertheless not isomorphic because there are 2γ associative elements in Dup_γ (Proposition 4.1.3) against only γ in Dendr_γ. Then, the free γ-multiplicial algebra over one generator is constructed (Theorem 4.1.6). Its underlying vector space is the vector space of the γ-edge valued binary trees and is endowed with 2γ products, similar to the over and under products on binary trees of Loday and Ronco [LR02]. Next, by using almost the same tools as the ones used in Section 2, we propose a generalization TDendr_γ of the tridendriform operad TDendr [LR04], called γ-polytridendriform operad. The operad TDendr_γ is defined as the Koszul dual of the γ-pluritridendriform operad Trias_γ, introduced by the author in [Gir16]. We obtain a presentation of TDendr_γ (Theorem 4.2.1) and an expression for its Hilbert series (Proposition 4.2.2). The dimensions of TDendr_γ thus obtained lead to establish the fact that the bases of TDendr_γ are indexed by γ-edge valued Schröder trees, that are Schröder trees where internal edges are labelled on $\{1, \dots, \gamma\}$. We end this work by providing generalizations on a nonnegative integer parameter γ integer generalization of all the operads intervening in the operadic butterfly. We then define the operads Com_γ, Lie_γ, Zin_γ, and Leib_γ, that are respective generalizations of the commutative operad, the Lie operad, the Zinbiel operad [Lod95] and the Leibniz operad [Lod93]. We provide analogous versions for our context of the arrows between the commutative operad and the Zinbiel operad (Proposition 4.3.1), and between the dendriform operad and the Zinbiel operad (Proposition 4.3.2).

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Notations and general conventions. All the algebraic structures of this article have a field of characteristic zero $\mathbb{K}$ as ground field. For any integers a and c, $[a, c]$ denotes the set $\{b \in \mathbb{N} : a \leqslant b \leqslant c\}$ and $[n]$, the set $[1, n]$. We use in all this paper the notations introduced in Section 1 of [Gir16].

1. Preliminaries: Koszul duality and the dendriform operad

In the present preliminary section, we will recall the notion of Koszul duality and several properties of the dendriform operad, the Koszul dual of the diassociative operad (see Section 1.3 of [Gir16]).

1.1. **Koszul duality.** In [GK94], Ginzburg and Kapranov extended the notion of Koszul duality of quadratic associative algebras to quadratic operads. Starting with a binary and quadratic operad $\mathcal{O}$ admitting a presentation $(\mathfrak{G}, \mathfrak{R})$, the *Koszul dual* of $\mathcal{O}$ is the operad $\mathcal{O}^!$, isomorphic to the operad admitting the presentation $\left(\mathfrak{G}, \mathfrak{R}^\perp\right)$ where $\mathfrak{R}^\perp$ is the annihilator of $\mathfrak{R}$ in $\mathbf{Free}(\mathfrak{G})$ with respect to the scalar product

$$\langle -, - \rangle : \mathbf{Free}(\mathfrak{G})(3) \otimes \mathbf{Free}(\mathfrak{G})(3) \to \mathbb{K} \tag{1.1.1}$$

linearly defined, for all $x, x', y, y' \in \mathfrak{G}(2)$, by

$$\langle x \circ_i y, x' \circ_{i'} y' \rangle := \begin{cases} 1 & \text{if } x = x', y = y', \text{ and } i = i' = 1, \\ -1 & \text{if } x = x', y = y', \text{ and } i = i' = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1.2}$$

Then, knowing a presentation of $\mathcal{O}$, one can compute a presentation of $\mathcal{O}^!$.

Furthermore, when $\mathcal{O}$ and $\mathcal{O}^!$ are two operads Koszul dual one of the other, and moreover, when they are Koszul operads and admit Hilbert series, their Hilbert series satisfy [GK94]

$$\mathcal{H}_{\mathcal{O}}\left(-\mathcal{H}_{\mathcal{O}^!}(-t)\right) = t. \tag{1.1.3}$$

We shall make use of (1.1.3) to compute the dimensions of Koszul operads defined as Koszul duals of known ones.

1.2. **Dendriform operad.** We recall here the definitions and some properties of the dendriform operad.

The *dendriform operad* Dendr was introduced by Loday [Lod01]. It is the operad admitting the presentation $(\mathfrak{G}_{\mathsf{Dendr}}, \mathfrak{R}_{\mathsf{Dendr}})$ where $\mathfrak{G}_{\mathsf{Dendr}} := \mathfrak{G}_{\mathsf{Dendr}}(2) := \{\prec, \succ\}$ and $\mathfrak{R}_{\mathsf{Dendr}}$ is the vector space generated by

$$\prec \circ_1 \succ - \succ \circ_2 \prec, \tag{1.2.1a}$$

$$\prec \circ_1 \prec - \prec \circ_2 \prec - \prec \circ_2 \succ, \tag{1.2.1b}$$

$$\succ \circ_1 \prec + \succ \circ_1 \succ - \succ \circ_2 \succ . \tag{1.2.1c}$$

Note that Dendr is a binary and quadratic operad.

This operad admits a quite complicated realization [Lod01]. For all $n \geqslant 1$, the $\mathsf{Dendr}(n)$ are vector spaces of binary trees with n internal nodes. The partial composition of two binary trees can be described by means of intervals of the Tamari order [HT72], a partial order relation involving binary trees. This realization shows that $\dim \mathsf{Dendr}(n) = \mathrm{cat}(n)$ where

$$\mathrm{cat}(n) := \frac{1}{n+1}\binom{2n}{n} \tag{1.2.2}$$

is the nth *Catalan number*, counting the binary trees with respect to their number of internal nodes. Therefore, the Hilbert series of Dendr satisfies

$$\mathcal{H}_{\mathsf{Dendr}}(t) = \frac{1 - \sqrt{1-4t} - 2t}{2t}. \tag{1.2.3}$$

Throughout this article, we shall graphically represent binary trees in a slightly different manner than syntax trees. We represent the leaves of binary trees by squares, internal nodes by circles, and edges by thick segments.

From the presentation of Dendr, we deduce that any Dendr-algebra, also called *dendriform algebra*, is a vector space $\mathcal{A}_{\mathsf{Dendr}}$ endowed with linear operations $\prec$ and $\succ$ satisfying the relations encoded by (1.2.1a)—(1.2.1c). Classical examples of dendriform algebras include Rota-Baxter algebras [Agu00] and shuffle algebras [Lod01].

The operation obtained by summing $\prec$ and $\succ$ is associative. Therefore, we can see a dendriform algebra as an associative algebra in which its associative product has been split into two parts satisfying Relations (1.2.1a), (1.2.1b), and (1.2.1c). More precisely, we say that an associative algebra $\mathcal{A}$ admits a *dendriform structure* if there exist two nonzero binary operations $\prec$ and $\succ$ such that the associative operation $\star$ of $\mathcal{A}$ satisfies $\star = \prec + \succ$, and $\mathcal{A}$ endowed with the operations $\prec$ and $\succ$, is a dendriform algebra

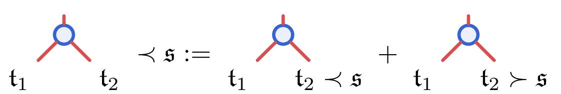
The free dendriform algebra $\mathcal{F}_{\mathsf{Dendr}}$ over one generator is the vector space Dendr of binary trees with at least one internal node endowed with the linear operations

$$\prec, \succ : \mathcal{F}_{\mathsf{Dendr}} \otimes \mathcal{F}_{\mathsf{Dendr}} \to \mathcal{F}_{\mathsf{Dendr}}, \tag{1.2.4}$$

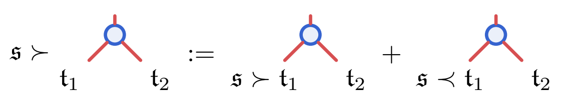
defined recursively, for any binary tree $\mathfrak{s}$ with at least one internal node, and binary trees $\mathfrak{t}_1$ and $\mathfrak{t}_2$ by

$$\mathfrak{s} \prec \; := \mathfrak{s} =: \; \succ \mathfrak{s}, \tag{1.2.5}$$

$$\prec \mathfrak{s} := 0 =: \mathfrak{s} \succ, \tag{1.2.6}$$



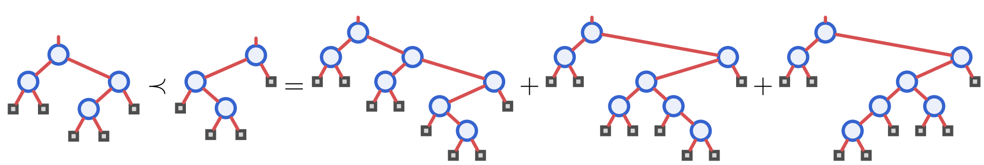
(1.2.7)



(1.2.8)

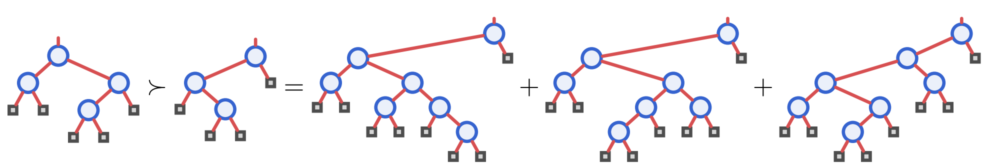
Note that neither $\prec$ nor $\succ$ are defined.

We have for instance,



(1.2.9)

and



(1.2.10)

As shown in [Lod01], the dendriform operad is the Koszul dual of the diassociative operad. This can be checked by a simple computation following what is explained in Section 1.1. Besides that, since theses two operads are Koszul operads, the alternating versions of their Hilbert series are the inverses for each other for series composition.

We invite the reader to take a look at [LR98, Agu00, Lod02, Foi07, EFMP08, EFM09, LV12] for a supplementary review of properties of dendriform algebras and of the dendriform operad.

2. Polydendriform operads

We introduce at this point our generalization on a nonnegative integer parameter γ of the dendriform operad and dendriform algebras. We first construct this operad, compute its dimensions, and give then two presentations by generators and relations. This section ends by a description of free algebras over one generator in the category encoded by our generalization.

2.1. Construction and properties.

Theorem 2.2.6 of [Gir16], by exhibiting a presentation of Dias_γ, shows that this operad is binary and quadratic. It then admits a Koszul dual, denoted by Dendr_γ and called *γ-polydendriform operad*.

2.1.1. *Definition and presentation.*

A description of Dendr_γ is provided by the following presentation by generators and relations.

Theorem 2.1.1. *For any integer $\gamma \geqslant 0$, the operad Dendr_γ admits the following presentation. It is generated by $\mathfrak{G}_{\mathsf{Dendr}_\gamma} := \mathfrak{G}_{\mathsf{Dendr}_\gamma}(2) := \{\leftharpoonup_a, \rightharpoonup_a : a \in [\gamma]\}$ and its space of relations $\mathfrak{R}_{\mathsf{Dendr}_\gamma}$ is generated by*

$$\leftharpoonup_a \circ_1 \rightharpoonup_{a'} - \rightharpoonup_{a'} \circ_2 \leftharpoonup_a, \qquad a, a' \in [\gamma], \tag{2.1.1a}$$

$$\leftharpoonup_a \circ_1 \leftharpoonup_b - \leftharpoonup_a \circ_2 \rightharpoonup_b, \qquad a < b \in [\gamma], \tag{2.1.1b}$$

$$\rightharpoonup_a \circ_1 \leftharpoonup_b - \rightharpoonup_a \circ_2 \rightharpoonup_b, \qquad a < b \in [\gamma], \tag{2.1.1c}$$

$$\leftharpoonup_a \circ_1 \leftharpoonup_b - \leftharpoonup_a \circ_2 \leftharpoonup_b, \qquad a < b \in [\gamma], \tag{2.1.1d}$$

$$\rightharpoonup_a \circ_1 \rightharpoonup_b - \rightharpoonup_a \circ_2 \rightharpoonup_b, \qquad a < b \in [\gamma], \tag{2.1.1e}$$

$$\leftharpoonup_d \circ_1 \leftharpoonup_d - \left(\sum_{c \in [d]} \leftharpoonup_d \circ_2 \leftharpoonup_c + \leftharpoonup_d \circ_2 \rightharpoonup_c \right), \qquad d \in [\gamma], \tag{2.1.1f}$$

$$\left(\sum_{c \in [d]} \rightharpoonup_d \circ_1 \rightharpoonup_c + \rightharpoonup_d \circ_1 \leftharpoonup_c \right) - \rightharpoonup_d \circ_2 \rightharpoonup_d, \qquad d \in [\gamma]. \tag{2.1.1g}$$

Proof. By Theorem 2.2.6 of [Gir16], we know that Dias_γ is a binary and quadratic operad, and that its space of relations $\mathfrak{R}_{\mathsf{Dias}_\gamma}$ is the space induced by the equivalence relation $\leftrightarrow_\gamma$ defined by (2.2.11a)–(2.2.11g) in [Gir16]. Now, by a straightforward computation, and by identifying $\leftharpoonup_a$ (resp. $\rightharpoonup_a$) with $\dashv_a$ (resp. $\vdash_a$) for any $a \in [\gamma]$, we obtain that the space $\mathfrak{R}_{\mathsf{Dendr}_\gamma}$ of the statement of the theorem satisfies $\mathfrak{R}_{\mathsf{Dias}_\gamma}^{\perp} = \mathfrak{R}_{\mathsf{Dendr}_\gamma}$. Hence, Dendr_γ admits the claimed presentation. □

Theorem 2.1.1 provides a quite complicated presentation of Dendr_γ. We shall below define a more convenient basis for the space of relations of Dendr_γ.

2.1.2. *Elements and dimensions.*

Proposition 2.1.2. *For any integer $\gamma \geqslant 0$, the Hilbert series $\mathcal{H}_{\mathsf{Dendr}_\gamma}(t)$ of the operad Dendr_γ satisfies*

$$\mathcal{H}_{\mathsf{Dendr}_\gamma}(t) = t + 2\gamma t\, \mathcal{H}_{\mathsf{Dendr}_\gamma}(t) + \gamma^2 t\, \mathcal{H}_{\mathsf{Dendr}_\gamma}(t)^2. \tag{2.1.2}$$

Proof. By setting $\bar{\mathcal{H}}_{\mathsf{Dendr}_\gamma}(t) := \mathcal{H}_{\mathsf{Dendr}_\gamma}(-t)$, from (2.1.2), we obtain

$$t = \frac{-\bar{\mathcal{H}}_{\mathsf{Dendr}_\gamma}(t)}{\left(1 + \gamma\, \bar{\mathcal{H}}_{\mathsf{Dendr}_\gamma}(t)\right)^2}. \tag{2.1.3}$$

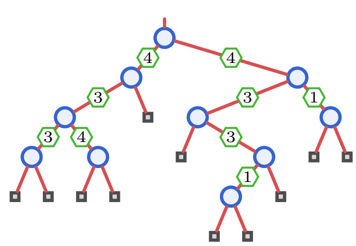
Moreover, by setting $\bar{\mathcal{H}}_{\mathsf{Dias}_\gamma}(t) := \mathcal{H}_{\mathsf{Dias}_\gamma}(-t)$, where $\mathcal{H}_{\mathsf{Dias}_\gamma}(t)$ is the Hilbert series of Dias_γ defined by (2.1.8) in [Gir16], we have

$$\bar{\mathcal{H}}_{\mathsf{Dias}_\gamma}\left(\bar{\mathcal{H}}_{\mathsf{Dendr}_\gamma}(t)\right) = \frac{-\bar{\mathcal{H}}_{\mathsf{Dendr}_\gamma}(t)}{\left(1 + \gamma\, \bar{\mathcal{H}}_{\mathsf{Dendr}_\gamma}(t)\right)^2} = t, \tag{2.1.4}$$

showing that $\bar{\mathcal{H}}_{\mathsf{Dias}_\gamma}(t)$ and $\bar{\mathcal{H}}_{\mathsf{Dendr}_\gamma}(t)$ are the inverses for each other for series composition.

Now, since by Theorem 2.3.1 and Proposition 2.1.1 of [Gir16], Dias_γ is a Koszul operad and its Hilbert series is $\mathcal{H}_{\mathsf{Dias}_\gamma}(t)$, and since Dendr_γ is by definition the Koszul dual of Dias_γ, the Hilbert series of these two operads satisfy (1.1.3). Therefore, (2.1.4) implies that the Hilbert series of Dendr_γ is $\mathcal{H}_{\mathsf{Dendr}_\gamma}(t)$. □

By examining the expression for $\mathcal{H}_{\mathsf{Dendr}_\gamma}(t)$ of the statement of Proposition 2.1.2, we observe that for any $n \geqslant 1$, $\mathsf{Dendr}_\gamma(n)$ can be seen as the vector space $\mathcal{F}_{\mathsf{Dendr}_\gamma}(n)$ of binary trees with n internal nodes wherein its $n-1$ edges connecting two internal nodes are labeled on $[\gamma]$. We call these trees *γ-edge valued binary trees.* In our graphical representations of γ-edge valued binary trees, any edge label is drawn into a hexagon located half the edge. For instance,



(2.1.5)

is a 4-edge valued binary tree and a basis element of $\mathsf{Dendr}_4(10)$.

We deduce from Proposition 2.1.2 that the Hilbert series of Dendr_γ satisfies

$$\mathcal{H}_{\mathsf{Dendr}_\gamma}(t) = \frac{1 - \sqrt{1 - 4\gamma t} - 2\gamma t}{2\gamma^2 t}, \tag{2.1.6}$$

and we also obtain that for all $n \geqslant 1$, $\dim \mathsf{Dendr}_\gamma(n) = \gamma^{n-1}\mathrm{cat}(n)$. For instance, the first dimensions of Dendr_1, Dendr_2, Dendr_3, and Dendr_4 are respectively

$$1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \tag{2.1.7}$$

$$1, 4, 20, 112, 672, 4224, 27456, 183040, 1244672, 8599552, 60196864, \tag{2.1.8}$$

$$1, 6, 45, 378, 3402, 32076, 312741, 3127410, 31899582, 330595668, 3471254514, \tag{2.1.9}$$

$$1, 8, 80, 896, 10752, 135168, 1757184, 23429120, 318636032, 4402970624, 61641588736. \tag{2.1.10}$$

The first one is Sequence **A000108**, the second one is Sequence **A003645**, and the third one is Sequence **A101600** of [Slo]. Last sequence is not listed in [Slo] at this time.

2.1.3. *Associative operations.* In the same manner as in the dendriform operad the sum of its two operations produces an associative operation, in the γ-dendriform operad there is a way to build associative operations, as shows next statement.

Proposition 2.1.3. *For any integers $\gamma \geqslant 0$ and $b \in [\gamma]$, the element*

$$\bullet_b := \pi\left(\sum_{a \in [b]} \leftharpoonup_a + \rightharpoonup_a\right) \tag{2.1.11}$$

of Dendr_γ, where $\pi : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dendr}_\gamma}\right) \to \mathsf{Dendr}_\gamma$ is the canonical surjection map, is associative.

Proof. By setting

$$x := \sum_{a \in [b]} \leftharpoonup_a + \rightharpoonup_a, \tag{2.1.12}$$

we have

$$\begin{aligned} x \circ_1 x - x \circ_2 x = & \leftharpoonup_a \circ_1 \leftharpoonup_{a'} + \leftharpoonup_a \circ_1 \rightharpoonup_{a'} + \rightharpoonup_a \circ_1 \leftharpoonup_{a'} + \rightharpoonup_a \circ_1 \rightharpoonup_{a'} \\ & - \leftharpoonup_a \circ_2 \leftharpoonup_{a'} - \leftharpoonup_a \circ_2 \rightharpoonup_{a'} - \rightharpoonup_a \circ_2 \leftharpoonup_{a'} - \rightharpoonup_a \circ_2 \rightharpoonup_{a'} . \end{aligned} \tag{2.1.13}$$

We the observe that (2.1.13) is the sum of elements (2.1.1a)—(2.1.1g) which generate, by Theorem 2.1.1, the space of relations of Dendr_γ. Therefore, we have $\pi(x \circ_1 x - x \circ_2 x) = 0$, implying $\bullet_b \circ_1 \bullet_b - \bullet_b \circ_2 \bullet_b = 0$ and the associativity of $\bullet_b$. $\square$

2.1.4. *Alternative presentation.* For any integer $\gamma \geqslant 0$, let $\prec_b$ and $\succ_b$, $b \in [\gamma]$, the elements of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dendr}_\gamma}\right)$ defined by

$$\prec_b := \sum_{a \in [b]} \leftharpoonup_a, \tag{2.1.14a}$$

and

$$\succ_b := \sum_{a \in [b]} \rightharpoonup_a . \tag{2.1.14b}$$

Then, since for all $b \in [\gamma]$ we have

$$\leftharpoonup_b = \begin{cases} \prec_1 & \text{if } b = 1, \\ \prec_b - \prec_{b-1} & \text{otherwise,} \end{cases} \tag{2.1.15a}$$

and

$$\rightharpoonup_b = \begin{cases} \succ_1 & \text{if } b = 1, \\ \succ_b - \succ_{b-1} & \text{otherwise,} \end{cases} \tag{2.1.15b}$$

by triangularity, the family $\mathfrak{G}'_{\mathsf{Dendr}_\gamma} := \{\prec_b, \succ_b \colon b \in [\gamma]\}$ forms a basis of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dendr}_\gamma}\right)(2)$ and then, generates $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dendr}_\gamma}\right)$ as an operad. This change of basis from $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dendr}_\gamma}\right)$ to $\mathbf{Free}(\mathfrak{G}'_{\mathsf{Dendr}_\gamma})$ is similar to the change of basis from $\mathbf{Free}(\mathfrak{G}'_{\mathsf{Dias}_\gamma})$ to $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ introduced in Section 2.3.6 of [Gir16]. Let us now express a presentation of Dendr_γ through the family $\mathfrak{G}'_{\mathsf{Dendr}_\gamma}$.

Theorem 2.1.4. *For any integer $\gamma \geqslant 0$, the operad Dendr_γ admits the following presentation. It is generated by $\mathfrak{G}'_{\mathsf{Dendr}_\gamma}$ and its space of relations $\mathfrak{R}'_{\mathsf{Dendr}_\gamma}$ is generated by*

$$\prec_a \circ_1 \succ_{a'} - \succ_{a'} \circ_2 \prec_a, \qquad a, a' \in [\gamma], \tag{2.1.16a}$$

$$\prec_a \circ_1 \prec_b - \prec_a \circ_2 \succ_b - \prec_a \circ_2 \prec_a, \qquad a < b \in [\gamma], \tag{2.1.16b}$$

$$\succ_a \circ_1 \succ_a + \succ_a \circ_1 \prec_b - \succ_a \circ_2 \succ_b, \qquad a < b \in [\gamma], \tag{2.1.16c}$$

$$\prec_b \circ_1 \prec_a - \prec_a \circ_2 \prec_b - \prec_a \circ_2 \succ_a, \qquad a < b \in [\gamma], \tag{2.1.16d}$$

$$\succ_a \circ_1 \prec_a + \succ_a \circ_1 \succ_b - \succ_b \circ_2 \succ_a, \qquad a < b \in [\gamma], \tag{2.1.16e}$$

$$\prec_a \circ_1 \prec_a - \prec_a \circ_2 \succ_a - \prec_a \circ_2 \prec_a, \qquad a \in [\gamma], \tag{2.1.16f}$$

$$\succ_a \circ_1 \succ_a + \succ_a \circ_1 \prec_a - \succ_a \circ_2 \succ_a, \qquad a \in [\gamma]. \tag{2.1.16g}$$

Proof. Let us show that $\mathfrak{R}'_{\mathsf{Dendr}_\gamma}$ is equal to the space of relations $\mathfrak{R}_{\mathsf{Dendr}_\gamma}$ of Dendr_γ defined in the statement of Theorem 2.1.1. By this last theorem, for any $x \in \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dendr}_\gamma}\right)(3)$, x is in $\mathfrak{R}_{\mathsf{Dendr}_\gamma}$ if and only if $\pi(x) = 0$ where $\pi : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dendr}_\gamma}\right) \to \mathsf{Dendr}_\gamma$ is the canonical surjection map. By straightforward computations, by expanding any element x of (2.1.16a)—(2.1.16g) over the elements $\leftharpoonup_a$, $\rightharpoonup_a$, $a \in [\gamma]$, by using (2.1.14a) and (2.1.14b) we obtain that x can be expressed as a sum of elements of $\mathfrak{R}_{\mathsf{Dendr}_\gamma}$. This implies that $\pi(x) = 0$ and hence that $\mathfrak{R}'_{\mathsf{Dendr}_\gamma}$ is a subspace of $\mathfrak{R}_{\mathsf{Dendr}_\gamma}$.

Now, one can observe that elements (2.1.16a)—(2.1.16f) are linearly independent. Then, $\mathfrak{R}'_{\mathsf{Dendr}_\gamma}$ has dimension $3\gamma^2$ which is also, by Theorem 2.1.1, the dimension of $\mathfrak{R}_{\mathsf{Dendr}_\gamma}$. The statement of the theorem follows. $\square$

The presentation of Dendr_γ provided by Theorem 2.1.4 is easier to handle than the one provided by Theorem 2.1.1. The main reason is that Relations (2.1.1f) and (2.1.1g) of the first presentation involve a nonconstant number of terms, while all relations of this second presentation always involve only two or three terms. As a very remarkable fact, it is worthwhile to note that the presentation of Dendr_γ provided by Theorem 2.1.4 can be directly obtained by considering the Koszul dual of Dias_γ over the K-basis (see Sections 2.3.5 and 2.3.6 of [Gir16]). Therefore, an alternative way to establish this presentation consists in computing the Koszul dual of Dias_γ seen through the presentation having $\mathfrak{R}'_{\mathsf{Dendr}_\gamma}$ as space of relations, which is made of the relations of Dias_γ expressed over the K-basis (see Proposition 2.3.8 of [Gir16]).

From now on, $\downarrow$ denotes the operation min on integers. Using this notation, the space of relations $\mathfrak{R}'_{\mathsf{Dendr}_\gamma}$ of Dendr_γ exhibited by Theorem 2.1.4 can be rephrased in a more compact way as the space generated by

$$\prec_a \circ_1 \succ_{a'} - \succ_{a'} \circ_2 \prec_a, \qquad a, a' \in [\gamma], \tag{2.1.17a}$$

$$\prec_a \circ_1 \prec_{a'} - \prec_{a \downarrow a'} \circ_2 \prec_a - \prec_{a \downarrow a'} \circ_2 \succ_{a'}, \qquad a, a' \in [\gamma], \tag{2.1.17b}$$

$$\succ_{a \downarrow a'} \circ_1 \prec_{a'} + \succ_{a \downarrow a'} \circ_1 \succ_a - \succ_a \circ_2 \succ_{a'}, \qquad a, a' \in [\gamma]. \tag{2.1.17c}$$

Over the family $\mathfrak{G}'_{\mathsf{Dendr}_\gamma}$, one can build associative operations in Dendr_γ in the following way.

Proposition 2.1.5. *For any integers $\gamma \geqslant 0$ and $b \in [\gamma]$, the element*

$$\odot_b := \pi(\prec_b + \succ_b) \tag{2.1.18}$$

of Dendr_γ, where $\pi : \mathbf{Free}(\mathfrak{G}'_{\mathsf{Dendr}_\gamma}) \to \mathsf{Dendr}_\gamma$ is the canonical surjection map, is associative.

Proof. By definition of the $\prec_b$ and $\succ_b$, $b \in [\gamma]$, we have

$$\prec_b + \succ_b = \sum_{a \in [b]} \leftharpoonup_a + \rightharpoonup_a . \tag{2.1.19}$$

We hence observe that $\odot_b = \bullet_b$, where $\bullet_b$ is the element of Dendr_γ defined in the statement of Proposition 2.1.3. Hence, by this latter proposition, $\odot_b$ is associative. $\square$

Proposition 2.1.6. *For any integer $\gamma \geqslant 0$, any associative element of Dendr_γ is proportional to $\odot_b$ for a $b \in [\gamma]$.*

Proof. Let $\pi : \mathbf{Free}(\mathfrak{G}'_{\mathsf{Dendr}_\gamma}) \to \mathsf{Dendr}_\gamma$ be the canonical surjection map. Consider the element

$$x := \sum_{a \in [\gamma]} \alpha_a \prec_a + \beta_a \succ_a \tag{2.1.20}$$

of $\mathbf{Free}(\mathfrak{G}'_{\mathsf{Dendr}_\gamma})$, where $\alpha_a, \beta_a \in \mathbb{K}$ for all $a \in [\gamma]$, such that $\pi(x)$ is associative in Dendr_γ. Since we have $\pi(r) = 0$ for all elements r of $\mathfrak{R}'_{\mathsf{Dendr}_\gamma}$ (see (2.1.17a), (2.1.17b), and (2.1.17c)), the fact that $\pi(x \circ_1 x - x \circ_2 x) = 0$ implies the constraints

$$\begin{aligned} \alpha_a\, \beta_{a'} &= \beta_{a'}\, \alpha_a, && a, a' \in [\gamma], \\ \alpha_a\, \alpha_{a'} &= \alpha_{a \downarrow a'}\, \alpha_a = \alpha_{a \downarrow a'}\, \beta_{a'}, && a, a' \in [\gamma], \\ \beta_{a \downarrow a'}\, \alpha_{a'} &= \beta_{a \downarrow a'}\, \beta_a = \beta_a\, \beta_{a'}, && a, a' \in [\gamma], \end{aligned} \tag{2.1.21}$$

on the coefficients intervening in x. Moreover, since the syntax trees $\succ_b \circ_1 \succ_a$, $\succ_b \circ_1 \prec_a$, $\prec_b \circ_2 \prec_a$, and $\prec_b \circ_2 \succ_a$ do not appear in $\mathfrak{R}'_{\mathsf{Dendr}_\gamma}$ for all $a < b \in [\gamma]$, we have the further constraints

$$\begin{aligned} \beta_b\, \beta_a &= 0, && a < b \in [\gamma], \\ \beta_b\, \alpha_a &= 0, && a < b \in [\gamma], \\ \alpha_b\, \alpha_a &= 0, && a < b \in [\gamma], \\ \alpha_b\, \beta_a &= 0, && a < b \in [\gamma]. \end{aligned} \tag{2.1.22}$$

These relations imply that there are at most one $c \in [\gamma]$ and one $d \in [\gamma]$ such that $\alpha_c \neq 0$ and $\beta_d \neq 0$. In this case, these relations imply also that $c = d$, and $\alpha_c = \beta_c$. Therefore, x is of the form $x = \alpha_a \prec_a + \alpha_a \succ_a$ for an $a \in [\gamma]$, whence the statement of the proposition. □

2.2. **Category of polydendriform algebras and free objects.** The aim of this section is to describe the category of Dendr_γ-algebras and more particularly the free Dendr_γ-algebra over one generator.

2.2.1. *Polydendriform algebras.* We call *γ-polydendriform algebra* any Dendr_γ-algebra. From the presentation of Dendr_γ provided by Theorem 2.1.1, any γ-polydendriform algebra is a vector space endowed with linear operations $\leftharpoonup_a, \rightharpoonup_a$, $a \in [\gamma]$, satisfying the relations encoded by (2.1.1a)—(2.1.1g). By considering the presentation of Dendr_γ exhibited by Theorem 2.1.4, any γ-polydendriform algebra is a vector space endowed with linear operations $\prec_a, \succ_a$, $a \in [\gamma]$, satisfying the relations encoded by (2.1.17a)—(2.1.17c).

2.2.2. *Two ways to split associativity.* Like dendriform algebras, which offer a way to split an associative operation into two parts, γ-polydendriform algebras propose two ways to split associativity depending on its chosen presentation.

On the one hand, in a γ-polydendriform algebra $\mathcal{D}$ over the operations $\leftharpoonup_a$, $\rightharpoonup_a$, $a \in [\gamma]$, by Proposition 2.1.3, an associative operation $\bullet$ is split into the 2γ operations $\leftharpoonup_a$, $\rightharpoonup_a$, $a \in [\gamma]$, so that for all $x, y \in \mathcal{D}$,

$$x \bullet y = \sum_{a \in [\gamma]} x \leftharpoonup_a y + x \rightharpoonup_a y, \tag{2.2.1}$$

and all partial sums operations $\bullet_b$, $b \in [\gamma]$, satisfying

$$x \bullet_b y = \sum_{a \in [b]} x \leftharpoonup_a y + x \rightharpoonup_a x, \tag{2.2.2}$$

also are associative.

On the other hand, in a γ-polydendriform algebra over the operations $\prec_a$, $\succ_a$, $a \in [\gamma]$, by Proposition 2.1.5, several associative operations $\odot_a$, $a \in [\gamma]$, are each split into two operations $\prec_a$, $\succ_a$, $a \in [\gamma]$, so that for all $x, y \in \mathcal{D}$,

$$x \odot_a y = x \prec_a y + x \succ_a y. \tag{2.2.3}$$

Therefore, we can observe that γ-polydendriform algebras over the operations $\leftharpoonup_a$, $\rightharpoonup_a$, $a \in [\gamma]$, are adapted to study associative algebras (by splitting its single product in the way we have described above) while γ-polydendriform algebras over the operations $\prec_a$, $\succ_a$, $a \in [\gamma]$, are adapted to study vectors spaces endowed with several associative products (by splitting each one in the way we have described above). Algebras with several associative products will be studied in Section 3.

2.2.3. *Free polydendriform algebras.* From now, in order to simplify and make uniform next definitions, we consider that in any γ-edge valued binary tree $\mathfrak{t}$, all edges connecting internal nodes of $\mathfrak{t}$ with leaves are labeled by ∞. By convention, for all $a \in [\gamma]$, we have $a \downarrow \infty = a = \infty \downarrow a$.

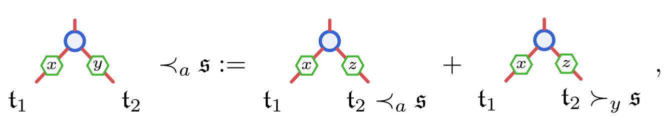
Let us endow the vector space $\mathcal{F}_{\mathsf{Dendr}_\gamma}$ of γ-edge valued binary trees with linear operations

$$\prec_a, \succ_a : \mathcal{F}_{\mathsf{Dendr}_\gamma} \otimes \mathcal{F}_{\mathsf{Dendr}_\gamma} \to \mathcal{F}_{\mathsf{Dendr}_\gamma}, \qquad a \in [\gamma], \tag{2.2.4}$$

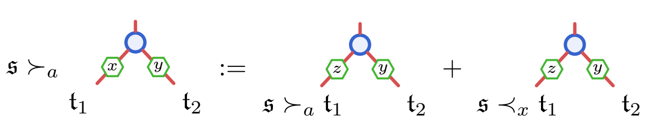
recursively defined, for any γ-edge valued binary tree $\mathfrak{s}$ and any γ-edge valued binary trees or leaves $\mathfrak{t}_1$ and $\mathfrak{t}_2$ by

$$\mathfrak{s} \prec_a \square := \mathfrak{s} =: \square \succ_a \mathfrak{s}, \tag{2.2.5}$$

$$\square \prec_a \mathfrak{s} := 0 =: \mathfrak{s} \succ_a \square, \tag{2.2.6}$$



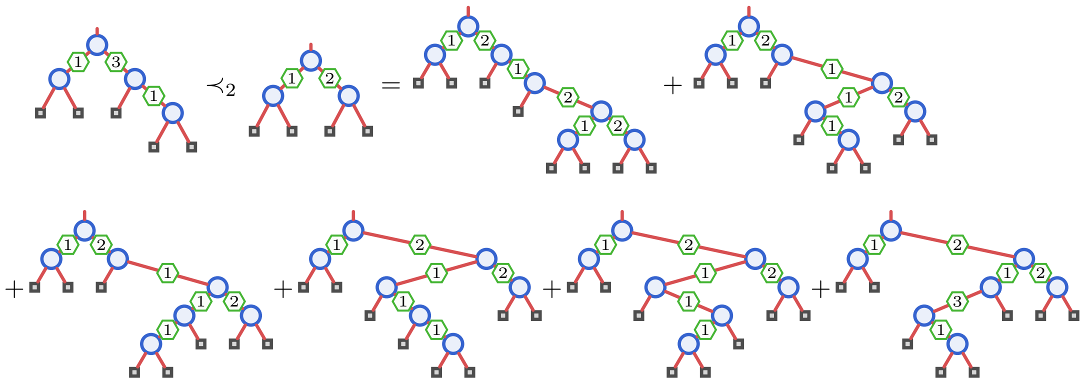
$$z := a \downarrow y, \tag{2.2.7}$$



$$z := a \downarrow x. \tag{2.2.8}$$

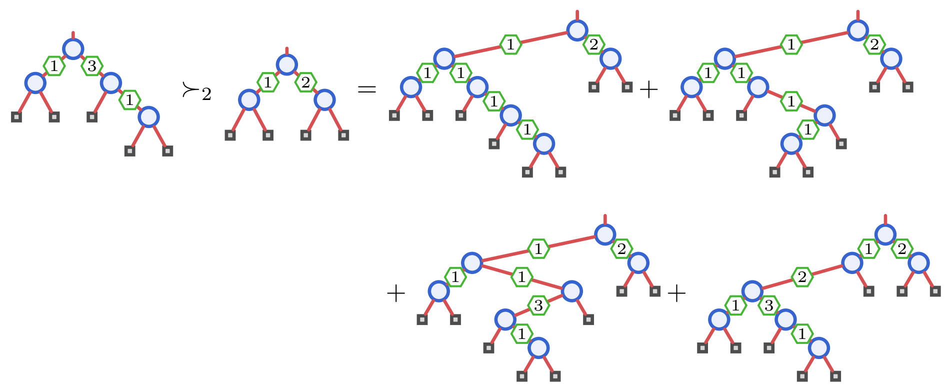
Note that neither $\square \prec_a \square$ nor $\square \succ_a \square$ are defined.

For example, we have



(2.2.9)

and

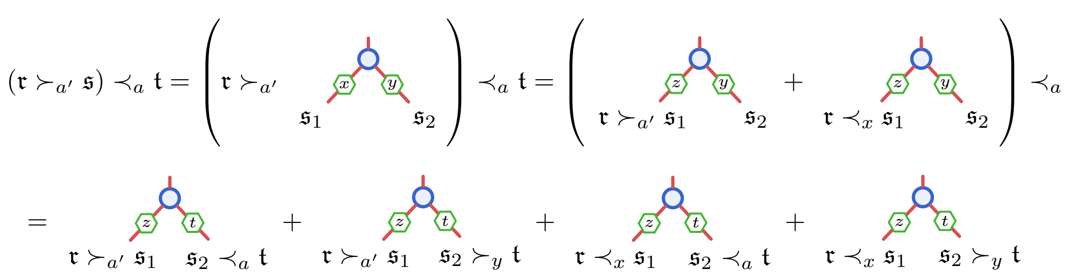


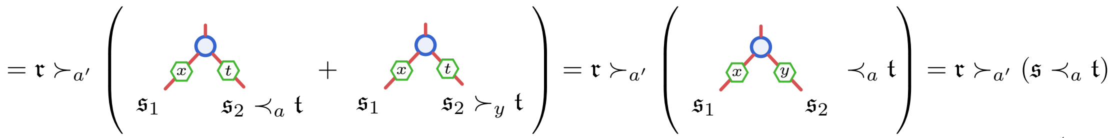
(2.2.10)

Lemma 2.2.1. *For any integer $\gamma \geqslant 0$, the vector space $\mathcal{F}_{\mathsf{Dendr}_\gamma}$ of γ-edge valued binary trees endowed with the operations $\prec_a$, $\succ_a$, $a \in [\gamma]$, is a γ-polydendriform algebra.*

Proof. We have to check that the operations $\prec_a$, $\succ_a$, $a \in [\gamma]$, of $\mathcal{F}_{\mathsf{Dendr}_\gamma}$ satisfy Relations (2.1.17a), (2.1.17b), and (2.1.17c) of γ-polydendriform algebras. Let $\mathfrak{r}$, $\mathfrak{s}$, and $\mathfrak{t}$ be three γ-edge valued binary trees and $a, a' \in [\gamma]$.

Denote by $\mathfrak{s}_1$ (resp. $\mathfrak{s}_2$) the left subtree (resp. right subtree) of $\mathfrak{s}$ and by x (resp. y) the label of the left (resp. right) edge incident to the root of $\mathfrak{s}$. We have

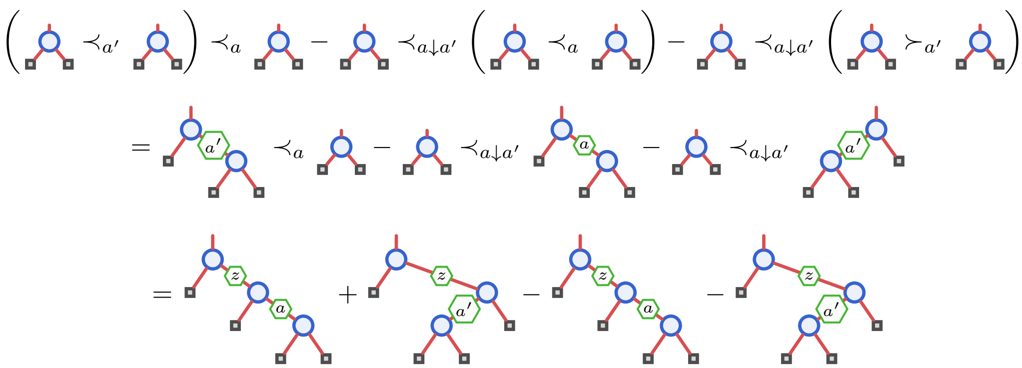




(2.2.11)

where $z := a' \downarrow x$ and $t := a \downarrow y$. This shows that (2.1.17a) is satisfied in $\mathcal{F}_{\mathsf{Dendr}_\gamma}$.

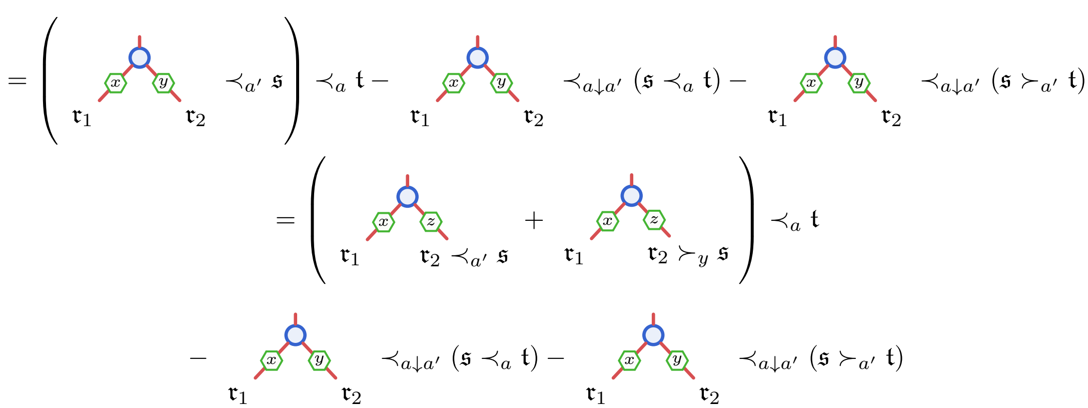
We now prove that Relations (2.1.17b) and (2.1.17c) hold by induction on the sum of the number of internal nodes of $\mathfrak{r}$, $\mathfrak{s}$, and $\mathfrak{t}$. Base case holds when all these trees have exactly one internal node, and since

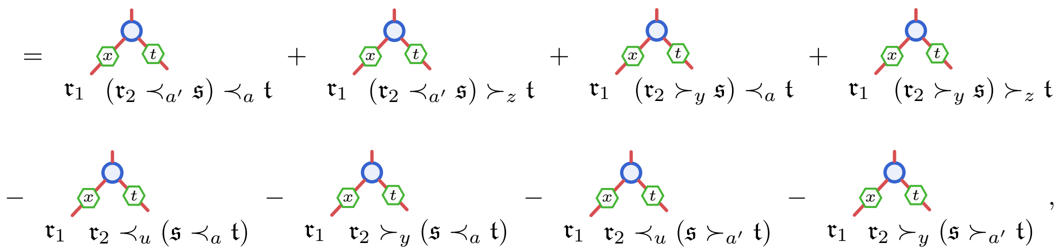


$= 0$, (2.2.12)

where $z := a \downarrow a'$, (2.1.17b) holds on trees with exactly one internal node. For the same arguments, we can show that (2.1.17c) holds on trees with exactly one internal node. Denote now by $\mathfrak{r}_1$ (resp. $\mathfrak{r}_2$) the left subtree (resp. right subtree) of $\mathfrak{r}$ and by x (resp. y) the label of the left (resp. right) edge incident to the root of $\mathfrak{r}$. We have

$$(\mathfrak{r} \prec_{a'} \mathfrak{s}) \prec_a \mathfrak{t} - \mathfrak{r} \prec_{a\downarrow a'} (\mathfrak{s} \prec_a \mathfrak{t}) - \mathfrak{r} \prec_{a\downarrow a'} (\mathfrak{s} \succ_{a'} \mathfrak{t})$$



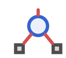


(2.2.13)

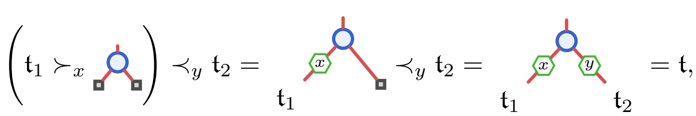
where $z := y \downarrow a'$, $t := z \downarrow a = y \downarrow a' \downarrow a$, and $u := a \downarrow a'$. Now, by induction hypothesis, Relation (2.1.17b) holds on $\mathfrak{r}_2$, $\mathfrak{s}$, and $\mathfrak{t}$. Hence, the sum of the first, fifth, and seventh terms of (2.2.13) is zero. Again by induction hypothesis, Relation (2.1.17c) holds on $\mathfrak{r}_2$, $\mathfrak{s}$, and $\mathfrak{t}$. Thus, the sum of the second, fourth, and last terms of (2.2.13) is zero. Finally, by what we just have proven in the first part of this proof, the sum of the third and sixth terms of (2.1.17c) is zero. Therefore, (2.2.13) is zero and (2.1.17b) is satisfied in $\mathcal{F}_{\mathsf{Dendr}_\gamma}$.

Finally, for the same arguments, we can show that (2.1.17c) is satisfied in $\mathcal{F}_{\mathsf{Dendr}_\gamma}$, implying the statement of the lemma. $\square$

Lemma 2.2.2. *For any integer $\gamma \geqslant 0$, the γ-pluriassociative algebra $\mathcal{F}_{\mathsf{Dendr}_\gamma}$ of γ-edge valued binary trees endowed with the operations $\prec_a$, $\succ_a$, $a \in [\gamma]$, is generated by*


. (2.2.14)

Proof. First, Lemma 2.2.1 shows that $\mathcal{F}_{\mathsf{Dendr}_\gamma}$ is a γ-polydendriform algebra. Let $\mathcal{D}$ be the γ-polydendriform subalgebra of $\mathcal{F}_{\mathsf{Dendr}_\gamma}$ generated by the tree (2.2.14). Let us show that any γ-edge valued binary tree $\mathfrak{t}$ is in $\mathcal{D}$ by induction on the number n of its internal nodes. When $n = 1$, $\mathfrak{t}$ = (2.2.14) and hence the property is satisfied. Otherwise, let $\mathfrak{t}_1$ (resp. $\mathfrak{t}_2$) be the left (resp. right) subtree of the root of $\mathfrak{t}$ and denote by x (resp. y) the label of the left (resp. right) edge incident to the root of $\mathfrak{t}$. Since $\mathfrak{t}_1$ and $\mathfrak{t}_2$ have less internal nodes than $\mathfrak{t}$, by induction hypothesis, $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are in $\mathcal{D}$. Moreover, by definition of the operations $\prec_a$, $\succ_a$, $a \in [\gamma]$, of $\mathcal{F}_{\mathsf{Dendr}_\gamma}$, one has



(2.2.15)

showing that $\mathfrak{t}$ also is in $\mathcal{D}$. Therefore, $\mathcal{D}$ is $\mathcal{F}_{\mathsf{Dendr}_\gamma}$, showing that $\mathcal{F}_{\mathsf{Dendr}_\gamma}$ is generated by the tree (2.2.14). $\square$

Theorem 2.2.3. *For any integer $\gamma \geqslant 0$, the vector space $\mathcal{F}_{\mathsf{Dendr}_\gamma}$ of γ-edge valued binary trees endowed with the operations $\prec_a$, $\succ_a$, $a \in [\gamma]$, is the free γ-polydendriform algebra over one generator.*

Proof. By Lemmas 2.2.1 and 2.2.2, $\mathcal{F}_{\mathsf{Dendr}_\gamma}$ is a γ-polydendriform algebra over one generator.

Moreover, since by Proposition 2.1.2, for any $n \geqslant 1$, the dimension of $\mathcal{F}_{\mathsf{Dendr}_\gamma}(n)$ is the same as the dimension of $\mathsf{Dendr}_\gamma(n)$, there cannot be relations in $\mathcal{F}_{\mathsf{Dendr}_\gamma}(n)$ involving $\mathfrak{g}$ that are not

γ-polydendriform relations (see (2.1.17a), (2.1.17b), and (2.1.17c)). Hence, $\mathcal{F}_{\mathsf{Dendr}_\gamma}$ is free as a γ-polydendriform algebra over one generator. $\square$

3. Multiassociative operads

There is a well-known diagram, whose definition is recalled below, gathering the diassociative, associative, and dendriform operads. The main goal of this section is to define a generalization on a nonnegative integer parameter of the associative operad to obtain a new version of this diagram, suited to the context of pluriassociative and polydendriform operads.

3.1. **Two generalizations of the associative operad.**

The associative operad is generated by one binary element. This operad admits two different generalizations generated by γ binary elements with the particularity that one is the Koszul dual of the other. We introduce and study in this section these two operads.

3.1.1. *Nonsymmetric associative operad.* Recall that the *nonsymmetric associative operad*, or the *associative operad* for short, is the operad As admitting the presentation $(\mathfrak{G}_{\mathsf{As}}, \mathfrak{R}_{\mathsf{As}})$, where $\mathfrak{G}_{\mathsf{As}} := \mathfrak{G}_{\mathsf{As}}(2) := \{\star\}$ and $\mathfrak{R}_{\mathsf{As}}$ is generated by $\star \circ_1 \star - \star \circ_2 \star$. It admits the following realization. For any $n \geqslant 1$, $\mathsf{As}(n)$ is the vector space of dimension one generated by the corolla of arity n and the partial composition $\mathfrak{c}_1 \circ_i \mathfrak{c}_2$ where $\mathfrak{c}_1$ is the corolla of arity n and $\mathfrak{c}_2$ is the corolla of arity m is the corolla of arity $n + m - 1$ for all valid i.

3.1.2. *Multiassociative operads.* For any integer $\gamma \geqslant 0$, we define As_γ as the operad admitting the presentation $\left(\mathfrak{G}_{\mathsf{As}_\gamma}, \mathfrak{R}_{\mathsf{As}_\gamma}\right)$, where $\mathfrak{G}_{\mathsf{As}_\gamma} := \mathfrak{G}_{\mathsf{As}_\gamma}(2) := \{\star_a : a \in [\gamma]\}$ and $\mathfrak{R}_{\mathsf{As}_\gamma}$ is generated by

$$\star_a \circ_1 \star_b - \star_b \circ_2 \star_b, \qquad a \leqslant b \in [\gamma], \tag{3.1.1a}$$

$$\star_b \circ_1 \star_a - \star_b \circ_2 \star_b, \qquad a < b \in [\gamma], \tag{3.1.1b}$$

$$\star_a \circ_2 \star_b - \star_b \circ_2 \star_b, \qquad a < b \in [\gamma], \tag{3.1.1c}$$

$$\star_b \circ_2 \star_a - \star_b \circ_2 \star_b, \qquad a < b \in [\gamma]. \tag{3.1.1d}$$

This space of relations can be rephrased in a more compact way as the space generated by

$$\star_a \circ_1 \star_{a'} - \star_{a \uparrow a'} \circ_2 \star_{a \uparrow a'}, \qquad a, a' \in [\gamma], \tag{3.1.2a}$$

$$\star_a \circ_2 \star_{a'} - \star_{a \uparrow a'} \circ_2 \star_{a \uparrow a'}, \qquad a, a' \in [\gamma]. \tag{3.1.2b}$$

We call As_γ the *γ-multiassociative operad*.

It follows immediately that As_γ is a set-operad and that it provides a generalization of the associative operad. The algebras over As_γ are the γ-multiassociative algebras introduced in Section 3.3.1 of [Gir16].

Let us now provide a realization of As_γ. A *γ-corolla* is a rooted tree with at most one internal node labeled on $[\gamma]$. Denote by $\mathcal{F}_{\mathsf{As}_\gamma}(n)$ the vector space of γ-corollas of arity $n \geqslant 1$, by $\mathcal{F}_{\mathsf{As}_\gamma}$ the graded vector space of all γ-corollas, and let

$$\star : \mathcal{F}_{\mathsf{As}_\gamma} \otimes \mathcal{F}_{\mathsf{As}_\gamma} \to \mathcal{F}_{\mathsf{As}_\gamma} \tag{3.1.3}$$

be the linear operation where, for any γ-corollas $\mathfrak{c}_1$ and $\mathfrak{c}_2$, $\mathfrak{c}_1 \star \mathfrak{c}_2$ is the γ-corolla with $n+m-1$ leaves and labeled by $a \uparrow a'$ where n (resp. m) is the number of leaves of $\mathfrak{c}_1$ (resp. $\mathfrak{c}_2$) and a (resp. a') is the label of $\mathfrak{c}_1$ (resp. $\mathfrak{c}_2$).

Proposition 3.1.1. *For any integer $\gamma \geqslant 0$, the operad As_γ is the vector space $\mathcal{F}_{\mathsf{As}_\gamma}$ of γ-corollas and its partial compositions satisfy, for any γ-corollas $\mathfrak{c}_1$ and $\mathfrak{c}_2$, $\mathfrak{c}_1 \circ_i \mathfrak{c}_2 = \mathfrak{c}_1 \star \mathfrak{c}_2$ for all valid integer i. Besides, As_γ is a Koszul operad and the set of right comb syntax trees of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{As}_\gamma}\right)$ where all internal nodes have a same label forms a Poincaré-Birkhoff-Witt basis of As_γ.*

Proof. In this proof, we consider that $\mathfrak{G}_{\mathsf{As}_\gamma}$ is totally ordered by the relation $\leqslant$ satisfying $\star_a \leqslant \star_b$ whenever $a \leqslant b \in [\gamma]$. It is immediate that the vector space $\mathcal{F}_{\mathsf{As}_\gamma}$ endowed with the partial compositions described in the statement of the proposition is an operad. Let us prove that this operad admits the presentation $\left(\mathfrak{G}_{\mathsf{As}_\gamma}, \mathfrak{R}_{\mathsf{As}_\gamma}\right)$.

For this purpose, consider the quadratic rewrite rule $\to_\gamma$ on $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{As}_\gamma}\right)$ satisfying

$$\star_a \circ_1 \star_b \to_\gamma \star_b \circ_2 \star_b, \qquad a \leqslant b \in [\gamma], \tag{3.1.4a}$$

$$\star_b \circ_1 \star_a \to_\gamma \star_b \circ_2 \star_b, \qquad a < b \in [\gamma], \tag{3.1.4b}$$

$$\star_a \circ_2 \star_b \to_\gamma \star_b \circ_2 \star_b, \qquad a < b \in [\gamma], \tag{3.1.4c}$$

$$\star_b \circ_2 \star_a \to_\gamma \star_b \circ_2 \star_b, \qquad a < b \in [\gamma]. \tag{3.1.4d}$$

Observe first that the space induced by the operad congruence induced by $\to_\gamma$ is $\mathfrak{R}_{\mathsf{As}_\gamma}$ (see (3.1.1a)—(3.1.1d)). Moreover, $\to_\gamma$ is a terminating rewrite rule and its normal forms are right comb syntax trees of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{As}_\gamma}\right)$ where all internal nodes have a same label. Besides, one can show that for any syntax tree $\mathfrak{t}$ of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{As}_\gamma}\right)$, we have $\mathfrak{t} \overset{*}{\to}_\gamma \mathfrak{s}$ with $\mathfrak{s}$ is a right comb syntax tree where all internal nodes labeled by the greatest label of $\mathfrak{t}$. Therefore, $\to_\gamma$ is a convergent rewrite rule and the operad As, admitting by definition the presentation $\left(\mathfrak{G}_{\mathsf{As}_\gamma}, \mathfrak{R}_{\mathsf{As}_\gamma}\right)$, has bases indexed by such trees.

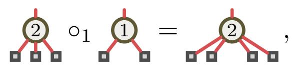
Now, let

$$\phi : \mathsf{As}_\gamma \simeq \mathbf{Free}\left(\mathfrak{G}_{\mathsf{As}_\gamma}\right) /_{\left\langle \mathfrak{R}_{\mathsf{As}_\gamma} \right\rangle} \to \mathcal{F}_{\mathsf{As}_\gamma} \tag{3.1.5}$$

be the map satisfying $\phi(\pi(\star_a)) = \mathfrak{c}_a$ where $\mathfrak{c}_a$ is the γ-corolla of arity 2 with internal node labeled by $a \in [\gamma]$ and $\pi : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{As}_\gamma}\right) \to \mathsf{As}_\gamma$ is the canonical surjection map. Since we have $\phi(\pi(x)) \circ_i \phi(\pi(y)) = \phi(\pi(x')) \circ_{i'} \phi(\pi(y'))$ for all relations $x \circ_i y \to_\gamma x' \circ_{i'} y'$ of (3.1.4a)—(3.1.4d), ϕ extends in a unique way into an operad morphism. First, since the set G_γ of all γ-corollas of arity two is a generating set of $\mathcal{F}_{\mathsf{As}_\gamma}$ and the image of ϕ contains G_γ, ϕ is surjective. Second, since by definition of $\mathcal{F}_{\mathsf{As}_\gamma}$, the bases of $\mathcal{F}_{\mathsf{As}_\gamma}$ are indexed by γ-corollas, in accordance with what we have shown in the previous paragraph of this proof, $\mathcal{F}_{\mathsf{As}_\gamma}$ and As_γ are isomorphic as graded vector spaces. Hence, ϕ is an operad isomorphism, showing that As_γ admits the claimed realization.

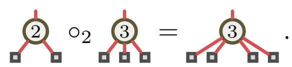
Finally, the existence of the convergent rewrite rule $\to_\gamma$ implies, by the Koszulity criterion [Hof10, DK10, LV12] we have reformulated in Section 1.2.5 of [Gir16], that As_γ is Koszul and that its Poincaré-Birkhoff-Witt basis is the one described in the statement of the proposition. □

We have for instance in As_3,



(3.1.6)

and



(3.1.7)

We deduce from Proposition 3.1.1 that the Hilbert series of As_γ satisfies

$$\mathcal{H}_{\mathsf{As}_\gamma}(t) = \frac{t + (\gamma - 1)t^2}{1 - t}. \tag{3.1.8}$$

and that for all $n \geqslant 2$, $\dim \mathsf{As}_\gamma(n) = \gamma$.

3.1.3. *Dual multiassociative operads.* Since As_γ is a binary and quadratic operad, its admits a Koszul dual, denoted by DAs_γ and called *γ-dual multiassociative operad*. The presentation of this operad is provided by next result.

Proposition 3.1.2. *For any integer $\gamma \geqslant 0$, the operad DAs_γ admits the following presentation. It is generated by $\mathfrak{G}_{\mathsf{DAs}_\gamma} := \mathfrak{G}_{\mathsf{DAs}_\gamma}(2) := \{\square_a : a \in [\gamma]\}$ and its space of relations $\mathfrak{R}_{\mathsf{DAs}_\gamma}$ is generated by*

$$\square_b \circ_1 \square_b - \square_b \circ_2 \square_b + \left(\sum_{a<b} \square_a \circ_1 \square_b + \square_b \circ_1 \square_a - \square_a \circ_2 \square_b - \square_b \circ_2 \square_a\right), \qquad b \in [\gamma]. \tag{3.1.9}$$

Proof. By a straightforward computation, and by identifying $\square_a$ with $\star_a$ for any $a \in [\gamma]$, we obtain that the space $\mathfrak{R}_{\mathsf{DAs}_\gamma}$ of the statement of the proposition satisfies $\mathfrak{R}_{\mathsf{DAs}_\gamma}^{\perp} = \mathfrak{R}_{\mathsf{As}_\gamma}$. Hence, DAs admits the claimed presentation. $\square$

For any integer $\gamma \geqslant 0$, let $\diamond_b$, $b \in [\gamma]$, the elements of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{DAs}_\gamma}\right)$ defined by

$$\diamond_b := \sum_{a \in [b]} \square_a. \tag{3.1.10}$$

Then, since for all $b \in [\gamma]$ we have

$$\square_b = \begin{cases} \diamond_1 & \text{if } b = 1, \\ \diamond_b - \diamond_{b-1} & \text{otherwise,} \end{cases} \tag{3.1.11}$$

by triangularity, the family $\mathfrak{G}'_{\mathsf{DAs}_\gamma} := \{\diamond_b : b \in [\gamma]\}$ forms a basis of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{DAs}_\gamma}\right)(2)$ and then, generates $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{DAs}_\gamma}\right)$ as an operad. Let us now express a presentation of DAs_γ through the family $\mathfrak{G}'_{\mathsf{DAs}_\gamma}$.

Proposition 3.1.3. *For any integer $\gamma \geqslant 0$, the operad DAs_γ admits the following presentation. It is generated by $\mathfrak{G}'_{\mathsf{DAs}_\gamma}$ and its space of relations $\mathfrak{R}'_{\mathsf{DAs}_\gamma}$ is generated by*

$$\diamond_a \circ_1 \diamond_a - \diamond_a \circ_2 \diamond_a, \qquad a \in [\gamma]. \tag{3.1.12}$$

Proof. Let us show that $\mathfrak{R}'_{\mathsf{DAs}_\gamma}$ is equal to the space of relations $\mathfrak{R}_{\mathsf{DAs}_\gamma}$ of DAs_γ defined in the statement of Proposition 3.1.2. By this last proposition, for any $x \in \mathbf{Free}\left(\mathfrak{G}_{\mathsf{DAs}_\gamma}\right)(3)$, x is in $\mathfrak{R}_{\mathsf{DAs}_\gamma}$ if and only if $\pi(x) = 0$ where $\pi : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{DAs}_\gamma}\right) \to \mathsf{DAs}$ is the canonical surjection map. By a straightforward computation, by expanding (3.1.12) over the elements $\boxtimes_a$, $a \in [\gamma]$, by using (3.1.10) we obtain that (3.1.12) can be expressed as a sum of elements of $\mathfrak{R}_{\mathsf{DAs}_\gamma}$. This implies that $\pi(x) = 0$ and hence that $\mathfrak{R}'_{\mathsf{DAs}_\gamma}$ is a subspace of $\mathfrak{R}_{\mathsf{DAs}_\gamma}$.

Now, one can observe that for all $a \in [\gamma]$, the elements (3.1.12) are linearly independent. Then, $\mathfrak{R}'_{\mathsf{DAs}_\gamma}$ has dimension γ which is also, by Proposition 3.1.2, the dimension of $\mathfrak{R}_{\mathsf{DAs}_\gamma}$. The statement of the proposition follows. □

Observe, from the presentation provided by Proposition 3.1.3 of DAs_γ, that DAs_2 is the operad denoted by *2as* in [LR06].

Notice that the Koszul dual of DAs_γ through its presentation $\left(\mathfrak{G}'_{\mathsf{DAs}_\gamma}, \mathfrak{R}'_{\mathsf{DAs}_\gamma}\right)$ of Proposition 3.1.3 gives rise to the following presentation for As_γ. This last operad admits the presentation $\left(\mathfrak{G}'_{\mathsf{As}_\gamma}, \mathfrak{R}'_{\mathsf{As}_\gamma}\right)$ where $\mathfrak{G}'_{\mathsf{As}_\gamma} := \mathfrak{G}'_{\mathsf{As}_\gamma}(2) := \{\triangle_a : a \in [\gamma]\}$ and $\mathfrak{R}'_{\mathsf{As}_\gamma}$ is generated by

$$\triangle_a \circ_1 \triangle_{a'}, \qquad a \neq a' \in [\gamma], \tag{3.1.13a}$$

$$\triangle_a \circ_2 \triangle_{a'}, \qquad a \neq a' \in [\gamma], \tag{3.1.13b}$$

$$\triangle_a \circ_1 \triangle_a - \triangle_a \circ_2 \triangle_a, \qquad a \in [\gamma]. \tag{3.1.13c}$$

Indeed, $\mathfrak{R}'_{\mathsf{As}_\gamma}$ is the space $\mathfrak{R}_{\mathsf{As}_\gamma}$ through the identification

$$\triangle_a = \begin{cases} \star_\gamma & \text{if } a = \gamma, \\ \star_a - \star_{a+1} & \text{otherwise.} \end{cases} \tag{3.1.14}$$

Proposition 3.1.4. *For any integer $\gamma \geqslant 0$, the Hilbert series $\mathcal{H}_{\mathsf{DAs}_\gamma}(t)$ of the operad DAs_γ satisfies*

$$\mathcal{H}_{\mathsf{DAs}_\gamma}(t) = t + t\,\mathcal{H}_{\mathsf{DAs}_\gamma}(t) + (\gamma - 1)\,\mathcal{H}_{\mathsf{DAs}_\gamma}(t)^2. \tag{3.1.15}$$

Proof. By setting $\bar{\mathcal{H}}_{\mathsf{DAs}_\gamma}(t) := \mathcal{H}_{\mathsf{DAs}_\gamma}(-t)$, from (3.1.15), we obtain

$$t = \frac{-\bar{\mathcal{H}}_{\mathsf{DAs}_\gamma}(t) + (\gamma - 1)\bar{\mathcal{H}}_{\mathsf{DAs}_\gamma}(t)^2}{1 + \bar{\mathcal{H}}_{\mathsf{DAs}_\gamma}(t)}. \tag{3.1.16}$$

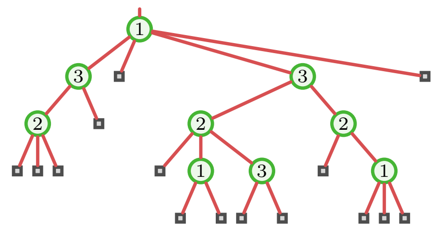
Moreover, by setting $\bar{\mathcal{H}}_{\mathsf{As}_\gamma}(t) := \mathcal{H}_{\mathsf{As}_\gamma}(-t)$, where $\mathcal{H}_{\mathsf{As}_\gamma}(t)$ is defined by (3.1.8), we have

$$\bar{\mathcal{H}}_{\mathsf{As}_\gamma}\left(\bar{\mathcal{H}}_{\mathsf{DAs}_\gamma}(t)\right) = \frac{-\bar{\mathcal{H}}_{\mathsf{DAs}_\gamma}(t) + (\gamma - 1)\bar{\mathcal{H}}_{\mathsf{DAs}_\gamma}(t)^2}{1 + \bar{\mathcal{H}}_{\mathsf{DAs}_\gamma}(t)} = t, \tag{3.1.17}$$

showing that $\bar{\mathcal{H}}_{\mathsf{As}_\gamma}(t)$ and $\bar{\mathcal{H}}_{\mathsf{DAs}_\gamma}(t)$ are the inverses for each other for series composition.

Now, since by Proposition 3.1.1, As_γ is a Koszul operad and its Hilbert series is $\mathcal{H}_{\mathsf{As}_\gamma}(t)$, and since DAs_γ is by definition the Koszul dual of As_γ, the Hilbert series of these two operads satisfy (1.1.3). Therefore, (3.1.17) implies that the Hilbert series of DAs_γ is $\mathcal{H}_{\mathsf{DAs}_\gamma}(t)$. □

A *Schröder tree* [Sta01, Sta11] is a planar rooted tree such that internal nodes have two of more children. By examining the expression for $\mathcal{H}_{\mathsf{DAs}_\gamma}(t)$ of the statement of Proposition 3.1.4, we observe that for any $n \geqslant 1$, $\mathsf{DAs}_\gamma(n)$ can be seen as the vector space $\mathcal{F}_{\mathsf{DAs}_\gamma}(n)$ of Schröder trees with n internal nodes, all labeled on $[\gamma]$ such that the label of an internal node is different from the labels of its children that are internal nodes. We call these trees *γ-alternating Schröder trees*. Let us also denote by $\mathcal{F}_{\mathsf{DAs}_\gamma}$ the graded vector space of all γ-alternating Schröder trees. For instance,



(3.1.18)

is a 3-alternating Schröder tree and a basis element of $\mathsf{DAs}_3(9)$.

We deduce also from Proposition 3.1.4 that

$$\mathcal{H}_{\mathsf{DAs}_\gamma}(t) = \frac{1 - \sqrt{1 - (4\gamma - 2)t + t^2} - t}{2(\gamma - 1)}. \tag{3.1.19}$$

By denoting by $\mathrm{nar}(n, k)$ the *Narayana number* [Nar55] defined by

$$\mathrm{nar}(n, k) := \frac{1}{k+1} \binom{n-1}{k} \binom{n}{k}, \tag{3.1.20}$$

we obtain that for all $n \geqslant 1$,

$$\dim \mathsf{DAs}_\gamma(n) = \sum_{k=0}^{n-2} \gamma^{k+1} (\gamma - 1)^{n-k-2} \,\mathrm{nar}(n-1, k). \tag{3.1.21}$$

This formula is a consequence of the fact that $\mathrm{nar}(n-1, k)$ is the number of binary trees with n leaves and with exactly k internal nodes having a internal node as a left child, the fact that the number $\mathrm{schr}(n)$ of Schröder trees with n leaves expresses as

$$\mathrm{schr}(n) = \sum_{k=0}^{n-2} 2^k \,\mathrm{nar}(n-1, k), \tag{3.1.22}$$

and the fact that any Schröder tree $\mathfrak{s}$ with n leaves can be encoded by a binary tree $\mathfrak{t}$ with n leaves where any left oriented edge connecting two internal nodes of $\mathfrak{t}$ is labeled on $[2]$ ($\mathfrak{s}$ is obtained from $\mathfrak{t}$ by contracting all edges labeled by 2).

For instance, the first dimensions of DAs_1, DAs_2, DAs_3, and DAs_4 are respectively

$$1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \tag{3.1.23}$$

$$1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098, 1037718, \tag{3.1.24}$$

$$1, 3, 15, 93, 645, 4791, 37275, 299865, 2474025, 20819307, 178003815, \tag{3.1.25}$$

$$1, 4, 28, 244, 2380, 24868, 272188, 3080596, 35758828, 423373636, 5092965724. \tag{3.1.26}$$

The second one is Sequence **A006318**, the third one is Sequence **A103210**, and the last one is Sequence **A103211** of [Slo].

Let us now establish a realization of DAs_γ.

Proposition 3.1.5. *For any nonnegative integer γ, the operad DAs_γ is the vector space $\mathcal{F}_{\mathsf{DAs}_\gamma}$ of γ-alternating Schröder trees. Moreover, for any γ-alternating Schröder trees $\mathfrak{s}$ and $\mathfrak{t}$, $\mathfrak{s} \circ_i \mathfrak{t}$ is the γ-alternating Schröder tree obtained by grafting the root of $\mathfrak{t}$ on the ith leaf x of $\mathfrak{s}$ and then, if the father y of x and the root z of $\mathfrak{t}$ have a same label, by contracting the edge connecting y and z.*

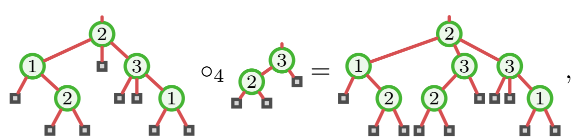
Proof. First, it is immediate that the vector space $\mathcal{F}_{\mathsf{DAs}_\gamma}$ endowed with the partial compositions described in the statement of the proposition is an operad.

Let

$$\phi : \mathsf{DAs}_\gamma \simeq \mathbf{Free}\left(\mathfrak{G}'_{\mathsf{DAs}_\gamma}\right) /_{\left\langle \mathfrak{R}'_{\mathsf{DAs}_\gamma} \right\rangle} \to \mathcal{F}_{\mathsf{DAs}_\gamma} \tag{3.1.27}$$

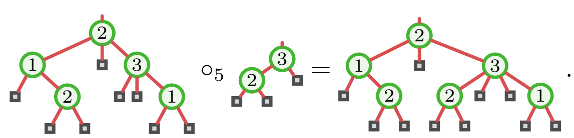
be the map satisfying $\phi(\pi(\diamond_a)) := \mathfrak{c}_a$ where $\mathfrak{c}_a$ is the γ-alternating Schröder with two leaves and one internal node labeled by $a \in [\gamma]$ and $\pi : \mathbf{Free}(\mathfrak{G}'_{\mathsf{DAs}_\gamma}) \to \mathsf{DAs}_\gamma$ is the canonical surjection map. Since we have $\phi(\pi(\diamond_a)) \circ_1 \phi(\pi(\diamond_a)) = \phi(\pi(\diamond_a)) \circ_2 \phi(\pi(\diamond_a))$ for all $a \in [\gamma]$, ϕ extends in a unique way into an operad morphism. First, since the set G_γ of all γ-alternating Schröder trees with two leaves and one internal node is a generating set of $\mathcal{F}_{\mathsf{DAs}_\gamma}$ and the image of ϕ contains G_γ, ϕ is surjective. Second, since by definition of $\mathcal{F}_{\mathsf{DAs}_\gamma}$, the bases of $\mathcal{F}_{\mathsf{DAs}_\gamma}$ are indexed by γ-alternating Schröder trees, by Proposition 3.1.4, $\mathcal{F}_{\mathsf{DAs}_\gamma}$ and DAs_γ are isomorphic as graded vector spaces. Hence, ϕ is an operad isomorphism, showing that DAs_γ admits the claimed realization. □

We have for instance in DAs_3,



(3.1.28)

and



(3.1.29)

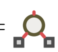
3.2. **A diagram of operads.** We now define morphisms between the operads Dias_γ, As_γ, DAs_γ, and Dendr_γ to obtain a generalization of a classical diagram involving the diassociative, associative, and dendriform operads.

3.2.1. *Relating the diassociative and dendriform operads.* The diagram

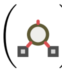
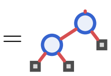
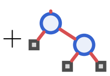
$$\begin{tikzcd} \mathsf{Dendr} & \mathsf{As} \arrow[l, "\zeta"'] \arrow[loop above, dotted, "!"] & \mathsf{Dias} \arrow[l, "\eta"'] \arrow[ll, bend right, dotted, "!"'] \end{tikzcd} \tag{3.2.1}$$

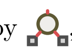
is a well-known diagram of operads, being a part of the so-called *operadic butterfly* [Lod01, Lod06] and summarizing in a nice way the links between the dendriform, associative, and diassociative operads. The operad As, being at the center of the diagram, is it own Koszul dual, while Dias and Dendr are Koszul dual one of the other.

The operad morphisms $\eta : \mathsf{Dias} \to \mathsf{As}$ and $\zeta : \mathsf{As} \to \mathsf{Dendr}$ are linearly defined through the realizations of Dias and Dendr recalled respectively in Section 1.3 of [Gir16] and in Section 1.2 by

$$\eta(\mathfrak{e}_{2,1}) :=$$  $$=: \eta(\mathfrak{e}_{2,2})\,, \tag{3.2.2}$$

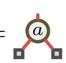
and

$$\zeta\left(\right.$$  $$\left.\right) :=$$  $$+$$  $$. \tag{3.2.3}$$

Since Dias is generated by $\mathfrak{e}_{2,1}$ and $\mathfrak{e}_{2,2}$, and since As is generated by , η and ζ are wholly defined.

3.2.2. *Relating the pluriassociative and polydendriform operads.*

Proposition 3.2.1. *For any integer $\gamma \geqslant 0$, the map $\eta_\gamma : \mathsf{Dias}_\gamma \to \mathsf{As}_\gamma$ satisfying*

$$\eta_\gamma(0a) =$$  $$= \eta_\gamma(a0), \qquad a \in [\gamma], \tag{3.2.4}$$

extends in a unique way into an operad morphism. Moreover, this morphism is surjective.

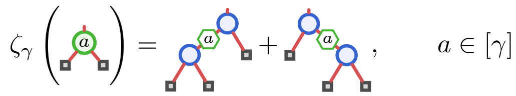
Proof. Theorem 2.2.6 of [Gir16] and Proposition 3.1.5 allow to interpret the map η_γ over the presentations of Dias_γ and As_γ. Then, via this interpretation, one has

$$\eta_\gamma(\pi(\dashv_a)) = \pi'(\star_a) = \eta_\gamma(\pi(\vdash_a)), \qquad a \in [\gamma], \tag{3.2.5}$$

where $\pi : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right) \to \mathsf{Dias}_\gamma$ and $\pi' : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{As}_\gamma}\right) \to \mathsf{As}_\gamma$ are canonical surjection maps. Now, for any element x of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ generating the space of relations $\mathfrak{R}_{\mathsf{Dias}_\gamma}$ of Dias_γ, we can check that $\eta_\gamma(\pi(x)) = 0$. This shows that η_γ extends in a unique way into an operad morphism. Finally, this morphism is a surjection since its image contains the set of all γ-corollas of arity 2, which is a generating set of As_γ. □

By Proposition 3.2.1, the map η_γ, whose definition is only given in arity 2, defines an operad morphism. Nevertheless, by induction on the arity, one can prove that for any word x of Dias_γ, $\eta_\gamma(x)$ is the γ-corolla of arity $|x|$ labeled by the greatest letter of x.

Proposition 3.2.2. *For any integer $\gamma \geqslant 0$, the map $\zeta_\gamma : \mathsf{DAs}_\gamma \to \mathsf{Dendr}_\gamma$ satisfying*



(3.2.6)

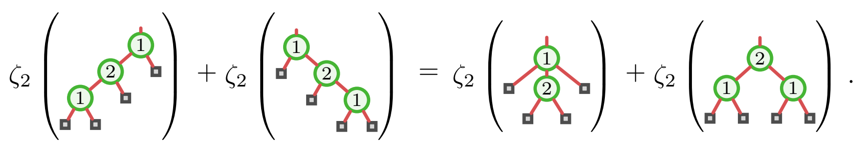
extends in a unique way into an operad morphism.

Proof. Propositions 3.1.3 and 3.1.5, and Theorem 2.1.4 allow to interpret the map ζ_γ over the presentations of DAs_γ and Dendr_γ. Then, via this interpretation, one has

$$\zeta_\gamma(\pi(\diamond_a)) = \pi'(\prec_a + \succ_a), \qquad a \in [\gamma], \tag{3.2.7}$$

where $\pi : \mathbf{Free}(\mathfrak{G}'_{\mathsf{DAs}_\gamma}) \to \mathsf{DAs}_\gamma$ and $\pi' : \mathbf{Free}\left(\mathfrak{G}'_{\mathsf{Dendr}_\gamma}\right) \to \mathsf{Dendr}_\gamma$ are canonical surjection maps. We now observe that the image of $\pi(\diamond_a)$ is $\odot_a$, where $\odot_a$ is the element of Dendr_γ defined in the statement of Proposition 2.1.5. Then, since by this last proposition this element is associative, for any element x of $\mathbf{Free}(\mathfrak{G}'_{\mathsf{DAs}_\gamma})$ generating the space of relations of $\mathfrak{R}'_{\mathsf{DAs}_\gamma}$ of DAs_γ, $\zeta_\gamma(\pi(x)) = 0$. This shows that ζ_γ extends in a unique way into an operad morphism. □

We have to observe that the morphism ζ_γ defined in the statement of Proposition 3.2.2 is injective only for $\gamma \leqslant 1$. Indeed, when $\gamma \geqslant 2$, we have the relation



(3.2.8)

Theorem 3.2.3. *For any integer $\gamma \geqslant 0$, the operads Dias_γ, Dendr_γ, As_γ, and DAs_γ fit into the diagram*

$$\begin{tikzcd}
\mathsf{Dendr}_\gamma & \mathsf{DAs}_\gamma \arrow[l, "\zeta_\gamma"'] \arrow[r, dotted, leftrightarrow, "!"] & \mathsf{As}_\gamma & \mathsf{Dias}_\gamma \arrow[l, "\eta_\gamma"'] \arrow[lll, dotted, leftrightarrow, bend right, "!"']
\end{tikzcd}, \tag{3.2.9}$$

where η_γ is the surjection defined in the statement of Proposition 3.2.1 and ζ_γ is the operad morphism defined in the statement of Proposition 3.2.2.

Proof. This is a direct consequence of Propositions 3.2.1 and 3.2.2. □

Diagram (3.2.9) is a generalization of (3.2.1) in which the associative operad split into operads As_γ and DAs_γ.

4. Further generalizations

In this last section, we propose some generalizations on a nonnegative integer parameter of well-known operads. For this, we use similar tools as the ones used in the first sections of this paper.

4.1. **Duplicial operad.** We construct here a generalization on a nonnegative integer parameter of the duplicial operad and describe the free algebras over one generator in the category encoded by this generalization.

4.1.1. *Multiplicial operads.* It is well-known [LV12] that the dendriform operad and the duplicial operad Dup [Lod08] are both specializations of a same operad D_q with one parameter $q \in \mathbb{K}$. This operad admits the presentation $\left(\mathfrak{G}_{\mathsf{D}_q}, \mathfrak{R}_{\mathsf{D}_q}\right)$, where $\mathfrak{G}_{\mathsf{D}_q} := \mathfrak{G}_{\mathsf{Dendr}}$ and $\mathfrak{R}_{\mathsf{D}_q}$ is the vector space generated by

$$\prec \circ_1 \succ - \succ \circ_2 \prec, \tag{4.1.1a}$$

$$\prec \circ_1 \prec - \prec \circ_2 \prec - q \prec \circ_2 \succ, \tag{4.1.1b}$$

$$q \succ \circ_1 \prec + \succ \circ_1 \succ - \succ \circ_2 \succ . \tag{4.1.1c}$$

One can observe that D_1 is the dendriform operad and that D_0 is the duplicial operad.

On the basis of this observation, from the presentation of Dendr_γ provided by Theorem 2.1.4 and its concise form provided by Relations (2.1.17a), (2.1.17b), and (2.1.17c) for its space of relations, we define the operad $\mathsf{D}_{q,\gamma}$ with two parameters, an integer $\gamma \geqslant 0$ and $q \in \mathbb{K}$, in the following way. We set $\mathsf{D}_{q,\gamma}$ as the operad admitting the presentation $\left(\mathfrak{G}_{\mathsf{D}_{q,\gamma}}, \mathfrak{R}_{\mathsf{D}_{q,\gamma}}\right)$, where $\mathfrak{G}_{\mathsf{D}_{q,\gamma}} := \mathfrak{G}'_{\mathsf{Dendr}_\gamma}$ and $\mathfrak{R}_{\mathsf{D}_{q,\gamma}}$ is the vector space generated by

$$\prec_a \circ_1 \succ_{a'} - \succ_{a'} \circ_2 \prec_a, \qquad a, a' \in [\gamma], \tag{4.1.2a}$$

$$\prec_a \circ_1 \prec_{a'} - \prec_{a \downarrow a'} \circ_2 \prec_a - q \prec_{a \downarrow a'} \circ_2 \succ_{a'}, \qquad a, a' \in [\gamma], \tag{4.1.2b}$$

$$q \succ_{a \downarrow a'} \circ_1 \prec_{a'} + \succ_{a \downarrow a'} \circ_1 \succ_a - \succ_a \circ_2 \succ_{a'}, \qquad a, a' \in [\gamma]. \tag{4.1.2c}$$

One can observe that $\mathsf{D}_{1,\gamma}$ is the operad Dendr_γ.

Let us define the operad Dup_γ, called *γ-multiplicial operad*, as the operad $\mathsf{D}_{0,\gamma}$. By using respectively the symbols $\hookleftarrow_a$ and $\hookrightarrow_a$ instead of $\prec_a$ and $\succ_a$ for all $a \in [\gamma]$, we obtain that the space of relations $\mathfrak{R}_{\mathsf{Dup}_\gamma}$ of Dup_γ is generated by

$$\hookleftarrow_a \circ_1 \hookrightarrow_{a'} - \hookrightarrow_{a'} \circ_2 \hookleftarrow_a, \qquad a, a' \in [\gamma], \tag{4.1.3a}$$

$$\hookleftarrow_a \circ_1 \hookleftarrow_{a'} - \hookleftarrow_{a \downarrow a'} \circ_2 \hookleftarrow_a, \qquad a, a' \in [\gamma], \tag{4.1.3b}$$

$$\hookrightarrow_{a \downarrow a'} \circ_1 \hookrightarrow_a - \hookrightarrow_a \circ_2 \hookrightarrow_{a'}, \qquad a, a' \in [\gamma]. \tag{4.1.3c}$$

We denote by $\mathfrak{G}_{\mathsf{Dup}_\gamma}$ the set of generators $\{\hookleftarrow_a, \hookrightarrow_a : a \in [\gamma]\}$ of Dup_γ.

In order to establish some properties of Dup_γ, let us consider the quadratic rewrite rule $\to_\gamma$ on $\mathbf{Free}(\mathfrak{G}_{\mathsf{Dup}_\gamma})$ satisfying

$$\hookleftarrow_a \circ_1 \hookrightarrow_{a'} \;\to_\gamma\; \hookrightarrow_{a'} \circ_2 \hookleftarrow_a, \qquad a, a' \in [\gamma], \tag{4.1.4a}$$

$$\hookleftarrow_a \circ_1 \hookleftarrow_{a'} \;\to_\gamma\; \hookleftarrow_{a \downarrow a'} \circ_2 \hookleftarrow_a, \qquad a, a' \in [\gamma], \tag{4.1.4b}$$

$$\hookrightarrow_a \circ_2 \hookrightarrow_{a'} \;\to_\gamma\; \hookrightarrow_{a \downarrow a'} \circ_1 \hookrightarrow_a, \qquad a, a' \in [\gamma]. \tag{4.1.4c}$$

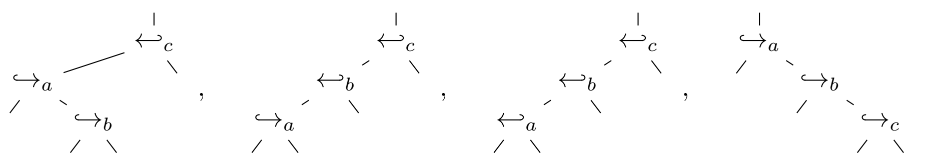
Observe that the space induced by the operad congruence induced by $\to_\gamma$ is $\mathfrak{R}_{\mathsf{Dup}_\gamma}$.

Lemma 4.1.1. *For any integer $\gamma \geqslant 0$, the rewrite rule $\to_\gamma$ is convergent and the generating series $\mathcal{G}_\gamma(t)$ of its normal forms counted by arity satisfies*

$$\mathcal{G}_\gamma(t) = t + 2\gamma t \, \mathcal{G}_\gamma(t) + \gamma^2 t \, \mathcal{G}_\gamma(t)^2. \tag{4.1.5}$$

Proof. Let us first prove that $\to_\gamma$ is terminating. Consider the map $\phi : \mathbf{Free}(\mathfrak{G}_{\mathsf{Dup}_\gamma}) \to \mathbb{N}^2$ defined, for any syntax tree $\mathfrak{t}$ by $\phi(\mathfrak{t}) := (\alpha + \alpha', \beta)$, where α (resp. α', β) is the sum, for all internal nodes of $\mathfrak{t}$ labeled by $\hookleftarrow_a$ (resp. $\hookrightarrow_a$, $\hookrightarrow_a$), $a \in [\gamma]$, of the number of internal nodes in its right (resp. left, right) subtree. For the lexicographical order $\leqslant$ on $\mathbb{N}^2$, we can check that for all $\to_\gamma$-rewritings $\mathfrak{s} \to_\gamma \mathfrak{t}$ where $\mathfrak{s}$ and $\mathfrak{t}$ are syntax trees with two internal nodes, we have $\phi(\mathfrak{s}) \neq \phi(\mathfrak{t})$ and $\phi(\mathfrak{s}) \leqslant \phi(\mathfrak{t})$. This implies that any syntax tree $\mathfrak{t}$ obtained by a sequence of $\to_\gamma$-rewritings from a syntax tree $\mathfrak{s}$ satisfies $\phi(\mathfrak{s}) \neq \phi(\mathfrak{t})$ and $\phi(\mathfrak{s}) \leqslant \phi(\mathfrak{t})$. Then, since the set of syntax trees of $\mathbf{Free}(\mathfrak{G}_{\mathsf{Dup}_\gamma})$ of a fixed arity is finite, this shows that $\to_\gamma$ is a terminating rewrite rule.

Let us now prove that $\to_\gamma$ is convergent. We call *critical tree* any syntax tree $\mathfrak{s}$ with three internal nodes that can be rewritten by $\to_\gamma$ into two different trees $\mathfrak{t}$ and $\mathfrak{t}'$. The pair $(\mathfrak{t}, \mathfrak{t}')$ is a *critical pair* for $\to_\gamma$. Critical trees for $\to_\gamma$ are, for all $a, b, c \in [\gamma]$,



(4.1.6)

Since $\to_\gamma$ is terminating, by the diamond lemma [New42] (see also [BN98]), to prove that $\to_\gamma$ is confluent, it is enough to check that for any critical tree $\mathfrak{s}$, there is a normal form $\mathfrak{r}$ of $\to_\gamma$ such that $\mathfrak{s} \to_\gamma \mathfrak{t} \overset{*}{\to}_\gamma \mathfrak{r}$ and $\mathfrak{s} \to_\gamma \mathfrak{t}' \overset{*}{\to}_\gamma \mathfrak{r}$, where $(\mathfrak{t}, \mathfrak{t}')$ is a critical pair. This can be done by hand for each of the critical trees depicted in (4.1.6).

Let us finally prove that the generating series of the normal forms of $\to_\gamma$ is (4.1.5). Since $\to_\gamma$ is terminating, its normal forms are the syntax trees that have no partial subtree equal to $\hookleftarrow_a \circ_1 \hookrightarrow_{a'}$, $\hookleftarrow_a \circ_1 \hookleftarrow_{a'}$, or $\hookrightarrow_a \circ_2 \hookrightarrow_{a'}$ for all $a, a' \in [\gamma]$. Then, the normal forms of $\to_\gamma$ are the syntax trees wherein any internal node labeled by $\hookleftarrow_a$, $a \in [\gamma]$, has a leaf as left child and any internal node labeled by $\hookrightarrow_a$, $a \in [\gamma]$, has a leaf or an internal node labeled by $\hookleftarrow_{a'}$, $a' \in [\gamma]$, as right child. Therefore, by denoting by $\mathcal{G}'_\gamma(t)$ the generating series of the normal forms of $\to_\gamma$ equal to the leaf or with a root labeled by $\hookleftarrow_a$, $a \in [\gamma]$, we obtain

$$\mathcal{G}'_\gamma(t) = t + \gamma t \, \mathcal{G}_\gamma(t) \tag{4.1.7}$$

and

$$\mathcal{G}_\gamma(t) = \mathcal{G}'_\gamma(t) + \gamma \, \mathcal{G}_\gamma(t) \mathcal{G}'_\gamma(t). \tag{4.1.8}$$

An elementary computation shows that $\mathcal{G}(t)$ satisfies (4.1.5). □

Proposition 4.1.2. *For any integer $\gamma \geqslant 0$, the operad Dup_γ is Koszul and for any integer $n \geqslant 1$, $\mathsf{Dup}_\gamma(n)$ is the vector space of γ-edge valued binary trees with n internal nodes.*

Proof. Since the space induced by the operad congruence induced by $\to_\gamma$ is $\mathfrak{R}_{\mathsf{Dup}_\gamma}$, and since by Lemma 4.1.1, $\to_\gamma$ is convergent, by the Koszulity criterion [Hof10, DK10, LV12] we have reformulated in Section 1.2.5 of [Gir16], Dup_γ is a Koszul operad. Moreover, again because $\to_\gamma$ is convergent, as a vector space, $\mathsf{Dup}_\gamma(n)$ is isomorphic to the vector space of the normal forms of $\to_\gamma$ with $n \geqslant 1$ internal nodes. Since the generating series $\mathcal{G}_\gamma(t)$ of the normal forms of $\rightarrow_\gamma$ is also the generating series of γ-edge valued binary trees (see Proposition 2.1.2), the second part of the statement of the proposition follows. $\square$

Since Proposition 4.1.2 shows that the operads Dup_γ and Dendr_γ have the same underlying vector space, asking if these two operads are isomorphic is natural. Next result implies that this is not the case.

Proposition 4.1.3. *For any integer $\gamma \geqslant 0$, any associative element of Dup_γ is proportional to $\pi(\leftarrow_a)$ or $\pi(\hookrightarrow_a)$ for an $a \in [\gamma]$, where $\pi : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dup}_\gamma}\right) \to \mathsf{Dup}_\gamma$ is the canonical surjection map.*

Proof. Let $\pi : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dup}_\gamma}\right) \to \mathsf{Dup}_\gamma$ be the canonical surjection map. Consider the element

$$x := \sum_{a \in [\gamma]} \alpha_a \leftarrow_a + \beta_a \hookrightarrow_a \tag{4.1.9}$$

of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dup}_\gamma}\right)$, where $\alpha_a, \beta_a \in \mathbb{K}$ for all $a \in [\gamma]$, such that $\pi(x)$ is associative in Dup_γ. Since we have $\pi(r) = 0$ for all elements r of $\mathfrak{R}_{\mathsf{Dup}_\gamma}$ (see (4.1.3a), (4.1.3b), and (4.1.3c)), the fact that $\pi(x \circ_1 x - x \circ_2 x) = 0$ implies the constraints

$$\begin{aligned} \alpha_a \, \beta_{a'} - \beta_{a'} \, \alpha_a &= 0, & a, a' &\in [\gamma], \\ \alpha_a \, \alpha_{a'} - \alpha_{a \downarrow a'} \, \alpha_a &= 0, & a, a' &\in [\gamma], \\ \beta_a \, \beta_{a'} - \beta_{a \downarrow a'} \, \beta_a &= 0, & a, a' &\in [\gamma], \end{aligned} \tag{4.1.10}$$

on the coefficients intervening in x. Moreover, since the syntax trees $\hookrightarrow_b \circ_1 \hookrightarrow_a$, $\hookrightarrow_a \circ_1 \leftarrow_{a'}$, $\leftarrow_b \circ_2 \leftarrow_a$, and $\leftarrow_a \circ_2 \hookrightarrow_{a'}$ do not appear in $\mathfrak{R}_{\mathsf{Dup}_\gamma}$ for all $a < b \in [\gamma]$ and $a, a' \in [\gamma]$, we have the further constraints

$$\begin{aligned} \beta_b \, \beta_a &= 0, & a < b &\in [\gamma], \\ \beta_a \, \alpha_{a'} &= 0, & a, a' &\in [\gamma], \\ \alpha_b \, \alpha_a &= 0, & a < b &\in [\gamma], \\ \alpha_a \, \beta_{a'} &= 0, & a, a' &\in [\gamma]. \end{aligned} \tag{4.1.11}$$

These relations imply that there are at most one $c \in [\gamma]$ and one $d \in [\gamma]$ such that $\alpha_c \neq 0$ and $\beta_d \neq 0$. In this case, the relations imply also that $\alpha_c = 0$ or $\beta_d = 0$, or both. Therefore, x is of the form $x = \alpha_a \leftarrow_a$ or $x = \beta_a \hookrightarrow_a$ for an $a \in [\gamma]$, whence the statement of the proposition. $\square$

By Proposition 4.1.3 there are exactly 2γ nonproportional associative operations in Dup_γ while, by Proposition 2.1.6 there are exactly γ such operations in Dendr_γ. Therefore, Dup_γ and Dendr_γ are not isomorphic.

4.1.2. *Free multiplicial algebras.* We call *γ-multiplicial algebra* any Dup_γ-algebra. From the definition of Dup_γ, any γ-multiplicial algebra is a vector space endowed with linear operations $\hookleftarrow_a, \hookrightarrow_a$, $a \in [\gamma]$, satisfying the relations encoded by (4.1.3a)—(4.1.3c).

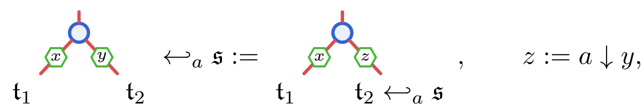
In order the simplify and make uniform next definitions, we consider that in any γ-edge valued binary tree $\mathfrak{t}$, all edges connecting internal nodes of $\mathfrak{t}$ with leaves are labeled by ∞. By convention, for all $a \in [\gamma]$, we have $a \downarrow \infty = a = \infty \downarrow a$. Let us endow the vector space $\mathcal{F}_{\mathsf{Dup}_\gamma}$ of γ-edge valued binary trees with linear operations

$$\hookleftarrow_a, \hookrightarrow_a : \mathcal{F}_{\mathsf{Dup}_\gamma} \otimes \mathcal{F}_{\mathsf{Dup}_\gamma} \to \mathcal{F}_{\mathsf{Dup}_\gamma}, \qquad a \in [\gamma], \tag{4.1.12}$$

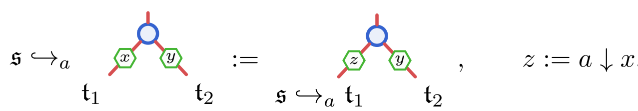
recursively defined, for any γ-edge valued binary tree $\mathfrak{s}$ and any γ-edge valued binary trees or leaves $\mathfrak{t}_1$ and $\mathfrak{t}_2$ by

$$\mathfrak{s} \hookleftarrow_a \square := \mathfrak{s} =: \square \hookrightarrow_a \mathfrak{s}, \tag{4.1.13}$$

$$\square \hookleftarrow_a \mathfrak{s} := 0 =: \mathfrak{s} \hookrightarrow_a \square, \tag{4.1.14}$$



(4.1.15)

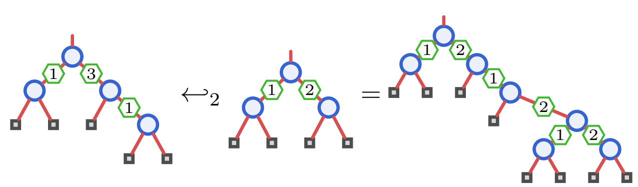


(4.1.16)

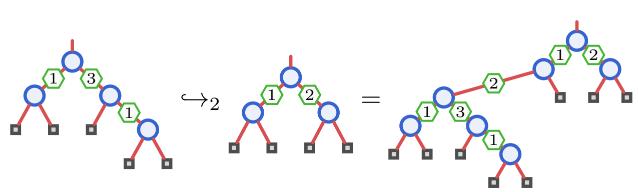
Note that neither $\square \prec_a \square$ nor $\square \hookrightarrow_a \square$ are defined.

These recursive definitions for the operations $\hookleftarrow_a$, $\hookrightarrow_a$, $a \in [\gamma]$, lead to the following direct reformulations. If $\mathfrak{s}$ and $\mathfrak{t}$ are two γ-edge valued binary trees, $\mathfrak{t} \hookleftarrow_a \mathfrak{s}$ (resp. $\mathfrak{s} \hookrightarrow_a \mathfrak{t}$) is obtained by replacing each label y (resp. x) of any edge in the rightmost (resp. leftmost) path of $\mathfrak{t}$ by $a \downarrow y$ (resp. $a \downarrow x$) to obtain a tree $\mathfrak{t}'$, and by grafting the root of $\mathfrak{s}$ on the rightmost (resp. leftmost) leaf of $\mathfrak{t}'$. These two operations are respective generalizations of the operations *under* and *over* on binary trees introduced by Loday and Ronco [LR02].

For example, we have


(4.1.17)

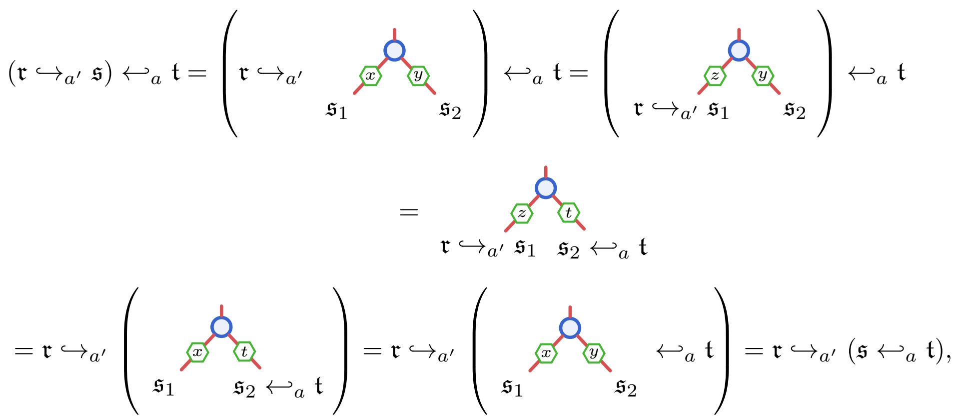
and


(4.1.18)

Lemma 4.1.4. *For any integer $\gamma \geqslant 0$, the vector space $\mathcal{F}_{\mathsf{Dup}_\gamma}$ of γ-edge valued binary trees endowed with the operations $\hookleftarrow_a$, $\hookrightarrow_a$, $a \in [\gamma]$, is a γ-multiplicial algebra.*

Proof. We have to check that the operations $\hookleftarrow_a$, $\hookrightarrow_a$, $a \in [\gamma]$, of $\mathcal{F}_{\mathsf{Dup}_\gamma}$ satisfy Relations (4.1.3a), (4.1.3b), and (4.1.3c) of γ-multiplicial algebras. Let $\mathfrak{r}$, $\mathfrak{s}$, and $\mathfrak{t}$ be three γ-edge valued binary trees and $a, a' \in [\gamma]$.

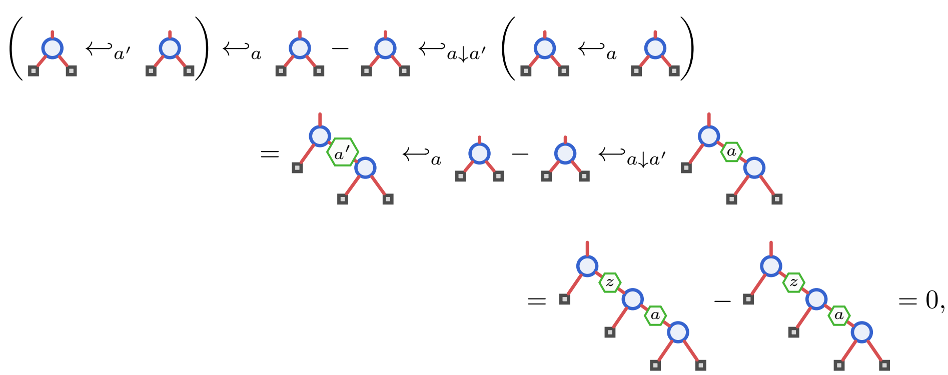
Denote by $\mathfrak{s}_1$ (resp. $\mathfrak{s}_2$) the left subtree (resp. right subtree) of $\mathfrak{s}$ and by x (resp. y) the label of the left (resp. right) edge incident to the root of $\mathfrak{s}$. We have



(4.1.19)

where $z := a' \downarrow x$ and $t := a \downarrow y$. This shows that (4.1.3a) is satisfied in $\mathcal{F}_{\mathsf{Dup}_\gamma}$.

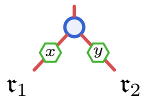
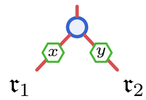
We now prove that Relations (4.1.3b) and (4.1.3c) hold by induction on the sum of the number of internal nodes of $\mathfrak{r}$, $\mathfrak{s}$, and $\mathfrak{t}$. Base case holds when all these trees have exactly one internal node, and since

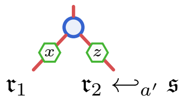
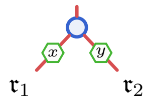


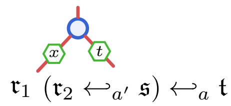
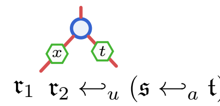
(4.1.20)

where $z := a \downarrow a'$, (4.1.3b) holds on trees with one internal node. For the same arguments, we can show that (4.1.3c) holds on trees with exactly one internal node. Denote now by $\mathfrak{r}_1$ (resp. $\mathfrak{r}_2$) the left subtree (resp. right subtree) of $\mathfrak{r}$ and by x (resp. y) the label of the left (resp. right) edge incident to the root of $\mathfrak{r}$. We have

$$(\mathfrak{r} \hookleftarrow_{a'} \mathfrak{s}) \hookleftarrow_a \mathfrak{t} - \mathfrak{r} \hookleftarrow_{a \downarrow a'} (\mathfrak{s} \hookleftarrow_a \mathfrak{t})$$

$$= \Bigg($$  $$\hookleftarrow_{a'} \mathfrak{s} \Bigg) \hookleftarrow_a \mathfrak{t} -$$  $$\hookleftarrow_{a \downarrow a'} (\mathfrak{s} \hookleftarrow_a \mathfrak{t})$$

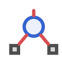
$$= \Bigg($$  $$\Bigg) \hookleftarrow_a \mathfrak{t} -$$  $$\hookleftarrow_{a \downarrow a'} (\mathfrak{s} \hookleftarrow_a \mathfrak{t})$$

$$=$$  $$-$$  $$, \qquad (4.1.21)$$

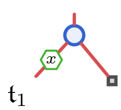
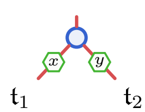
where $z := y \downarrow a'$, $t := z \downarrow a = y \downarrow a' \downarrow a$, and $u := a \downarrow a'$. Now, since by induction hypothesis Relation (4.1.3b) holds on $\mathfrak{r}_2$, $\mathfrak{s}$, and $\mathfrak{t}$, (4.1.21) is zero. Therefore, (4.1.3b) is satisfied in $\mathcal{F}_{\mathsf{Dup}_\gamma}$.

Finally, for the same arguments, we can show that (4.1.3c) is satisfied in $\mathcal{F}_{\mathsf{Dup}_\gamma}$, implying the statement of the lemma. □

Lemma 4.1.5. *For any integer $\gamma \geqslant 0$, the γ-multiplicial algebra $\mathcal{F}_{\mathsf{Dup}_\gamma}$ of γ-edge valued binary trees endowed with the operations $\hookleftarrow_a$, $\hookrightarrow_a$, $a \in [\gamma]$, is generated by*

 $$. \qquad (4.1.22)$$

Proof. First, Lemma 4.1.4 shows that $\mathcal{F}_{\mathsf{Dup}_\gamma}$ is a γ-multiplicial algebra. Let $\mathcal{M}$ be the γ-multiplicial subalgebra of $\mathcal{F}_{\mathsf{Dup}_\gamma}$ generated by . Let us show that any γ-edge valued binary tree $\mathfrak{t}$ is in $\mathcal{M}$ by induction on the number n of its internal nodes. When $n = 1$, $\mathfrak{t} =$  and hence the property is satisfied. Otherwise, let $\mathfrak{t}_1$ (resp. $\mathfrak{t}_2$) be the left (resp. right) subtree of the root of $\mathfrak{t}$ and denote by x (resp. y) the label of the left (resp. right) edge incident to the root of $\mathfrak{t}$. Since $\mathfrak{t}_1$ and $\mathfrak{t}_2$ have less internal nodes than $\mathfrak{t}$, by induction hypothesis, $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are in $\mathcal{M}$. Moreover, by definition of the operations $\hookleftarrow_a$, $\hookrightarrow_a$, $a \in [\gamma]$, of $\mathcal{F}_{\mathsf{Dup}_\gamma}$, one has

$$\Big(\mathfrak{t}_1 \hookrightarrow_x$$  $$\Big) \hookleftarrow_y \mathfrak{t}_2 =$$  $$\hookleftarrow_y \mathfrak{t}_2 =$$  $$= \mathfrak{t}, \qquad (4.1.23)$$

showing that $\mathfrak{t}$ also is in $\mathcal{M}$. Therefore, $\mathcal{M}$ is $\mathcal{F}_{\mathsf{Dup}_\gamma}$, showing that $\mathcal{F}_{\mathsf{Dup}_\gamma}$ is generated by . □

Theorem 4.1.6. *For any integer $\gamma \geqslant 0$, the vector space $\mathcal{F}_{\mathsf{Dup}_\gamma}$ of γ-valued binary trees endowed with the operations $\hookleftarrow_a$, $\hookrightarrow_a$, $a \in [\gamma]$, is the free γ-multiplicial algebra over one generator.*

Proof. By Lemmas 4.1.4 and 4.1.5, $\mathcal{F}_{\mathsf{Dup}_\gamma}$ is a γ-multiplicial algebra over one generator.

Moreover, since by Proposition 4.1.2, for any $n \geqslant 1$, the dimension of $\mathcal{F}_{\mathsf{Dup}_\gamma}(n)$ is the same as the dimension of $\mathsf{Dup}_\gamma(n)$, there cannot be relations in $\mathcal{F}_{\mathsf{Dup}_\gamma}(n)$ involving $\mathfrak{g}$ that are not γ-multiplicial relations (see (4.1.3a), (4.1.3b), and (4.1.3c)). Hence, $\mathcal{F}_{\mathsf{Dup}_\gamma}$ is free as a γ-multiplicial algebra over one generator. □

4.2. **Polytridendriform operads.** We propose here a generalization TDendr_γ on a nonnegative integer parameter γ of the tridendriform operad [LR04]. This last operad is the Koszul dual of the triassociative operad. We proceed by using an analogous strategy as the one used to define the operads Dendr_γ as Koszul duals of Dias_γ. Indeed, we define TDendr_γ as the Koszul dual of the operad Trias_γ, called γ-pluritriassociative operad, a generalization of the triassociative operad defined in [Gir16].

Since the proofs of the results contained in this section are very similar to the ones of Section 2, we omit proofs here.

Theorem 4.2.1 of [Gir16], by exhibiting a presentation of Trias_γ, shows that this operad is binary and quadratic. It then admits a Koszul dual, denoted by TDendr_γ and called *γ-polytridendriform operad.*

Theorem 4.2.1. *For any integer $\gamma \geqslant 0$, the operad TDendr_γ admits the following presentation. It is generated by $\mathfrak{G}_{\mathsf{TDendr}_\gamma} := \mathfrak{G}_{\mathsf{TDendr}_\gamma}(2) := \{\leftharpoonup_a, \wedge, \rightharpoonup_a : a \in [\gamma]\}$ and its space of relations $\mathfrak{R}_{\mathsf{TDendr}_\gamma}$ is generated by*

$$\wedge \circ_1 \wedge - \wedge \circ_2 \wedge, \tag{4.2.1a}$$

$$\leftharpoonup_a \circ_1 \wedge - \wedge \circ_2 \leftharpoonup_a, \qquad a \in [\gamma], \tag{4.2.1b}$$

$$\wedge \circ_1 \rightharpoonup_a - \rightharpoonup_a \circ_2 \wedge, \qquad a \in [\gamma], \tag{4.2.1c}$$

$$\wedge \circ_1 \leftharpoonup_a - \wedge \circ_2 \rightharpoonup_a, \qquad a \in [\gamma], \tag{4.2.1d}$$

$$\leftharpoonup_a \circ_1 \rightharpoonup_{a'} - \rightharpoonup_{a'} \circ_2 \leftharpoonup_a, \qquad a, a' \in [\gamma], \tag{4.2.1e}$$

$$\leftharpoonup_a \circ_1 \leftharpoonup_b - \leftharpoonup_a \circ_2 \rightharpoonup_b, \qquad a < b \in [\gamma], \tag{4.2.1f}$$

$$\rightharpoonup_a \circ_1 \leftharpoonup_b - \rightharpoonup_a \circ_2 \rightharpoonup_b, \qquad a < b \in [\gamma], \tag{4.2.1g}$$

$$\leftharpoonup_b \circ_1 \leftharpoonup_a - \leftharpoonup_a \circ_2 \leftharpoonup_b, \qquad a < b \in [\gamma], \tag{4.2.1h}$$

$$\rightharpoonup_a \circ_1 \rightharpoonup_b - \rightharpoonup_b \circ_2 \rightharpoonup_a, \qquad a < b \in [\gamma], \tag{4.2.1i}$$

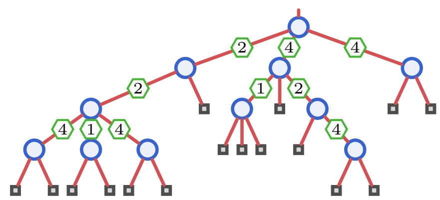
$$\leftharpoonup_d \circ_1 \leftharpoonup_d - \leftharpoonup_d \circ_2 \wedge - \left(\sum_{c \in [d]} \leftharpoonup_d \circ_2 \leftharpoonup_c + \leftharpoonup_d \circ_2 \rightharpoonup_c\right), \qquad d \in [\gamma], \tag{4.2.1j}$$

$$\left(\sum_{c \in [d]} \rightharpoonup_d \circ_1 \leftharpoonup_c + \rightharpoonup_d \circ_1 \rightharpoonup_c\right) + \rightharpoonup_d \circ_1 \wedge - \rightharpoonup_d \circ_2 \rightharpoonup_d, \qquad d \in [\gamma]. \tag{4.2.1k}$$

Proposition 4.2.2. *For any integer $\gamma \geqslant 0$, the Hilbert series $\mathcal{H}_{\mathsf{TDendr}_\gamma}(t)$ of the operad TDendr_γ satisfies*

$$\mathcal{H}_{\mathsf{TDendr}_\gamma}(t) = t + (2\gamma + 1)t\,\mathcal{H}_{\mathsf{TDendr}_\gamma}(t) + \gamma(\gamma + 1)t\,\mathcal{H}_{\mathsf{TDendr}_\gamma}(t)^2. \tag{4.2.2}$$

By examining the expression for $\mathcal{H}_{\mathsf{TDendr}_\gamma}(t)$ of the statement of Proposition 4.2.2, we observe that for any $n \geqslant 1$, $\mathsf{TDendr}(n)$ can be seen as the vector space $\mathcal{F}_{\mathsf{TDendr}_\gamma}(n)$ of Schröder trees with $n+1$ leaves wherein its edges connecting two internal nodes are labeled on $[\gamma]$. We call these trees *γ-edge valued Schröder trees*. For instance,



(4.2.3)

is a 4-edge valued Schröder tree and a basis element of $\mathsf{TDendr}_4(16)$.

We deduce from Proposition 4.2.2 that

$$\mathcal{H}_{\mathsf{TDendr}_\gamma}(t) = \frac{1 - \sqrt{1 - (4\gamma + 2)t + t^2} - (2\gamma + 1)t}{2(\gamma + \gamma^2)t}. \tag{4.2.4}$$

Moreover, we obtain that for all $n \geqslant 1$,

$$\dim \mathsf{TDendr}_\gamma(n) = \sum_{k=0}^{n-1} (\gamma + 1)^k \gamma^{n-k-1} \operatorname{nar}(n, k), \tag{4.2.5}$$

where $\operatorname{nar}(n, k)$ is defined in (3.1.20). For instance, the first dimensions of TDendr_1, TDendr_2, TDendr_3, and TDendr_4 are respectively

$$1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859, 2646723, \tag{4.2.6}$$

$$1, 5, 31, 215, 1597, 12425, 99955, 824675, 6939769, 59334605, 513972967, \tag{4.2.7}$$

$$1, 7, 61, 595, 6217, 68047, 770149, 8939707, 105843409, 1273241431, 15517824973, \tag{4.2.8}$$

$$1, 9, 101, 1269, 17081, 240849, 3511741, 52515549, 801029681, 12414177369, 194922521301. \tag{4.2.9}$$

The first one is Sequence **A001003** of [Slo]. The others sequences are not listed in [Slo] at this time.

4.3. **Operads of the operadic butterfly.** The *operadic butterfly* [Lod01,Lod06] is a diagram gathering seven famous operads. We have seen in Section 3.2 that this diagram gathers the diassociative, associative, and dendriform operads. It involves also the *commutative operad* Com, the *Lie operad* Lie, the *Zinbiel operad* Zin [Lod95], and the *Leibniz operad* Leib [Lod93]. It is of the form

$$\begin{tikzcd}
 & \mathsf{Dendr} \arrow[dl] \arrow[rr, dotted, leftrightarrow, "!"] & & \mathsf{Dias} \arrow[dl] & \\
\mathsf{Zin} \arrow[rrrr, dotted, leftrightarrow, bend right, "!"] & & \mathsf{As} \arrow[ul] \arrow[dl] \arrow[loop above, dotted, "!"] & & \mathsf{Leib} \arrow[ul] \arrow[dl] \\
 & \mathsf{Com} \arrow[ul] \arrow[rr, dotted, leftrightarrow, "!"] & & \mathsf{Lie} \arrow[ul] &
\end{tikzcd} \tag{4.3.1}$$

and as it shows, some operads are Koszul dual of some others (in particular, $\mathsf{Com}^! = \mathsf{Lie}$ and $\mathsf{Zin}^! = \mathsf{Leib}$).

We have to emphasize the fact the operads Com, Lie, Zin, and Leib of the operadic butterfly are symmetric operads. The computation of the Koszul dual of a symmetric operad does not follows what we have presented in Section 1.1. We invite the reader to consult [GK94] or [LV12] for a complete description.

For simplicity, in what follows, we shall consider algebras over symmetric operads instead of symmetric operads.

4.3.1. *A generalization of the operadic butterfly.* A possible continuation to this work consists in constructing a diagram

$$\begin{tikzcd}
& \mathsf{Dendr}_\gamma \arrow[dl] \arrow[rrr, leftrightarrow, dotted, "!"] & & & \mathsf{Dias}_\gamma \arrow[dl] & \\
\mathsf{Zin}_\gamma \arrow[rrrrr, leftrightarrow, dotted, bend right, "!"] & & \mathsf{DAs}_\gamma \arrow[ul] \arrow[dl] \arrow[r, leftrightarrow, dotted, "!"] & \mathsf{As}_\gamma & & \mathsf{Leib}_\gamma \arrow[ul] \arrow[dl] \\
& \mathsf{Com}_\gamma \arrow[ul] \arrow[rrr, leftrightarrow, dotted, "!"] & & & \mathsf{Lie}_\gamma \arrow[ul] &
\end{tikzcd} \tag{4.3.2}$$

where DAs_γ is the γ-dual multiassociative operad defined in Section 3.1.3 and Com_γ, Lie_γ, Zin_γ, and Leib_γ, respectively are generalizations on a nonnegative integer parameter γ of the operads Com, Lie, Zin, and Leib. Let us now define these operads.

4.3.2. *Commutative and Lie operads.* The symmetric operad Com is the symmetric operad describing the category of algebras $\mathcal{C}$ with one binary operation $\diamond$, subjected for any elements x, y, and z of $\mathcal{C}$ to the two relations

$$x \diamond y = y \diamond x, \tag{4.3.3a}$$

$$(x \diamond y) \diamond z = x \diamond (y \diamond z). \tag{4.3.3b}$$

This operad has the property to be a commutative version of $\mathsf{As} = \mathsf{DAs}_1$.

We define the symmetric operad Com_γ by using the same idea of being a commutative version of DAs_γ. Therefore, Com_γ is the symmetric operad describing the category of algebras $\mathcal{C}$ with binary operations $\diamond_a$, $a \in [\gamma]$, subjected for any elements x, y, and z of $\mathcal{C}$ to the two sorts of relations

$$x \diamond_a y = y \diamond_a x, \qquad a \in [\gamma], \tag{4.3.4a}$$

$$(x \diamond_a y) \diamond_a z = x \diamond_a (y \diamond_a z), \qquad a \in [\gamma]. \tag{4.3.4b}$$

Moreover, we define the symmetric operad Lie_γ as the Koszul dual of Com_γ.

4.3.3. *Zinbiel and Leibniz operads.* The symmetric operad Zin is the symmetric operad describing the category of algebras $\mathcal{Z}$ with one generating binary operation $\sqcup\!\sqcup$, subjected for any elements x, y, and z of $\mathcal{Z}$ to the relation

$$(x \sqcup\!\sqcup y) \sqcup\!\sqcup z = x \sqcup\!\sqcup (y \sqcup\!\sqcup z) + x \sqcup\!\sqcup (z \sqcup\!\sqcup y). \tag{4.3.5}$$

This operad has the property to be a commutative version of $\mathsf{Dendr} = \mathsf{Dendr}_1$. Indeed, Relation (4.3.5) is obtained from Relations (1.2.1a), (1.2.1b), and (1.2.1c) of dendriform algebras with the condition that for any elements x and y, $x \prec y = y \succ x$, and by setting $x \sqcup\!\sqcup y := x \prec y$.

We define the symmetric operad Zin_γ by using the same idea of having the property to be a commutative version of Dendr_γ. Therefore, Zin_γ is the symmetric operad describing the category of algebras $\mathcal{Z}$ with binary operations $\sqcup\!\sqcup_a$, $a \in [\gamma]$, subjected for any elements x, y, and z of $\mathcal{Z}$ to the relation

$$(x \sqcup\!\sqcup_{a'} y) \sqcup\!\sqcup_a z = x \sqcup\!\sqcup_{a \downarrow a'} (y \sqcup\!\sqcup_a z) + x \sqcup\!\sqcup_{a \downarrow a'} (z \sqcup\!\sqcup_{a'} y), \qquad a, a' \in [\gamma]. \tag{4.3.6}$$

Relation (4.3.6) is obtained from Relations (2.1.17a), (2.1.17b), and (2.1.17c) of γ-polydendriform algebras with the condition that for any elements x and y and $a \in [\gamma]$, $x \prec_a y = y \succ_a x$, and by setting $x \sqcup\!\sqcup_a y := x \prec_a y$. Moreover, we define the symmetric operad Leib_γ as the Koszul dual of Zin_γ.

Proposition 4.3.1. *For any integer $\gamma \geqslant 0$ and any Zin_γ-algebra $\mathcal{Z}$, the binary operations $\diamond_a$, $a \in [\gamma]$, defined for all elements x and y of $\mathcal{Z}$ by*

$$x \diamond_a y := x \sqcup\!\sqcup_a y + y \sqcup\!\sqcup_a x, \qquad a \in [\gamma], \tag{4.3.7}$$

endow $\mathcal{Z}$ with a Com_γ-algebra structure.

Proof. Since for all $a \in [\gamma]$ and all elements x and y of $\mathcal{Z}$, by (4.3.6), we have

$$x \diamond_a y - y \diamond_a x = x \sqcup\!\sqcup_a y + y \sqcup\!\sqcup_a x - y \sqcup\!\sqcup_a x - x \sqcup\!\sqcup_a y = 0, \tag{4.3.8}$$

the operations $\diamond_a$ satisfy Relation (4.3.4a) of Com_γ-algebras. Moreover, since for all $a \in [\gamma]$ and all elements x, y, and z of $\mathcal{Z}$, by (4.3.6), we have

$$\begin{aligned}
&(x \diamond_a y) \diamond_a z - x \diamond_a (y \diamond_a z) \\
&\qquad = (x \sqcup\!\sqcup_a y + y \sqcup\!\sqcup_a x) \sqcup\!\sqcup_a z + z \sqcup\!\sqcup_a (x \sqcup\!\sqcup_a y + y \sqcup\!\sqcup_a x) \\
&\qquad\quad - x \sqcup\!\sqcup_a (y \sqcup\!\sqcup_a z + z \sqcup\!\sqcup_a y) - (y \sqcup\!\sqcup_a z + z \sqcup\!\sqcup_a y) \sqcup\!\sqcup_a x \\
&\qquad = (x \sqcup\!\sqcup_a y) \sqcup\!\sqcup_a z + (y \sqcup\!\sqcup_a x) \sqcup\!\sqcup_a z + z \sqcup\!\sqcup_a (x \sqcup\!\sqcup_a y) + z \sqcup\!\sqcup_a (y \sqcup\!\sqcup_a x) \\
&\qquad\quad - x \sqcup\!\sqcup_a (y \sqcup\!\sqcup_a z) - x \sqcup\!\sqcup_a (z \sqcup\!\sqcup_a y) - (y \sqcup\!\sqcup_a z) \sqcup\!\sqcup_a x - (z \sqcup\!\sqcup_a y) \sqcup\!\sqcup_a x \\
&\qquad = (y \sqcup\!\sqcup_a x) \sqcup\!\sqcup_a z - (y \sqcup\!\sqcup_a z) \sqcup\!\sqcup_a x \\
&\qquad = y \sqcup\!\sqcup_a (x \sqcup\!\sqcup_a z) + y \sqcup\!\sqcup_a (z \sqcup\!\sqcup_a x) - y \sqcup\!\sqcup_a (z \sqcup\!\sqcup_a x) - y \sqcup\!\sqcup_a (x \sqcup\!\sqcup_a z) \\
&\qquad = 0,
\end{aligned} \tag{4.3.9}$$

the operations $\diamond_a$ satisfy Relation (4.3.4b) of Com_γ-algebras. Hence, $\mathcal{Z}$ is a Com_γ-algebra. □

Proposition 4.3.2. *For any integer $\gamma \geqslant 0$, and any Zin_γ-algebra $\mathcal{Z}$, the binary operations $\prec_a$, $\succ_a$, $a \in [\gamma]$ defined for all elements x and y of $\mathcal{Z}$ by*

$$x \prec_a y := x \sqcup\!\sqcup_a y, \qquad a \in [\gamma], \tag{4.3.10}$$

and

$$x \succ_a y := y \sqcup\!\sqcup_a x, \qquad a \in [\gamma], \tag{4.3.11}$$

endow $\mathcal{Z}$ with a γ-polydendriform algebra structure.

Proof. Since, for all $a, a' \in [\gamma]$ and all elements x, y, and z of $\mathcal{Z}$, by (4.3.6), we have

$$\begin{aligned}(x \succ_{a'} y) \prec_a z - x \succ_{a'} (y \prec_a z) \\ &= (y \sqcup\!\sqcup_{a'} x) \sqcup\!\sqcup_a z - (y \sqcup\!\sqcup_a z) \sqcup\!\sqcup_{a'} x \\ &= y \sqcup\!\sqcup_{a \downarrow a'} (x \sqcup\!\sqcup_a z) + y \sqcup\!\sqcup_{a \downarrow a'} (z \sqcup\!\sqcup_{a'} x) - y \sqcup\!\sqcup_{a \downarrow a'} (z \sqcup\!\sqcup_{a'} x) - y \sqcup\!\sqcup_{a \downarrow a'} (x \sqcup\!\sqcup_a z) \\ &= 0,\end{aligned} \tag{4.3.12}$$

the operations $\prec_a$ and $\succ_a$ satisfy Relation (2.1.17a) of γ-polydendriform algebras. Moreover, since for all $a, a' \in [\gamma]$ and all elements x, y, and z of $\mathcal{Z}$, by (4.3.6), we have

$$\begin{aligned}(x \prec_{a'} y) \prec_a z - x \prec_{a \downarrow a'} (y \prec_a z) - x \prec_{a \downarrow a'} (y \succ_{a'} z) \\ &= (x \sqcup\!\sqcup_{a'} y) \sqcup\!\sqcup_a z - x \sqcup\!\sqcup_{a \downarrow a'} (y \sqcup\!\sqcup_a z) - x \sqcup\!\sqcup_{a \downarrow a'} (z \sqcup\!\sqcup_{a'} y) \\ &= x \sqcup\!\sqcup_{a \downarrow a'} (y \sqcup\!\sqcup_a z) + x \sqcup\!\sqcup_{a \downarrow a'} (z \sqcup\!\sqcup_{a'} y) - x \sqcup\!\sqcup_{a \downarrow a'} (y \sqcup\!\sqcup_a z) - x \sqcup\!\sqcup_{a \downarrow a'} (z \sqcup\!\sqcup_{a'} y) \\ &= 0,\end{aligned} \tag{4.3.13}$$

the operations $\prec_a$ and $\succ_a$ satisfy Relation (2.1.17b) of γ-polydendriform algebras. Finally, since for all $a, a' \in [\gamma]$ and all elements x, y, and z of $\mathcal{Z}$, we have

$$\begin{aligned}(x \prec_{a'} y) \succ_{a \downarrow a'} z + (x \succ_a y) \succ_{a \downarrow a'} z - x \succ_a (y \succ_{a'} z) \\ &= z \sqcup\!\sqcup_{a \downarrow a'} (x \sqcup\!\sqcup_{a'} y) + z \sqcup\!\sqcup_{a \downarrow a'} (y \sqcup\!\sqcup_a x) - (z \sqcup\!\sqcup_{a'} y) \sqcup\!\sqcup_a x \\ &= z \sqcup\!\sqcup_{a \downarrow a'} (x \sqcup\!\sqcup_{a'} y) + z \sqcup\!\sqcup_{a \downarrow a'} (y \sqcup\!\sqcup_a x) - z \sqcup\!\sqcup_{a \downarrow a'} (y \sqcup\!\sqcup_a x) - z \sqcup\!\sqcup_{a \downarrow a'} (x \sqcup\!\sqcup_{a'} y) \\ &= 0,\end{aligned} \tag{4.3.14}$$

the operations $\prec_a$ and $\succ_a$ satisfy Relation (2.1.17c) of γ-polydendriform algebras. Hence $\mathcal{Z}$ is a γ-polydendriform algebra. □

The constructions stated by Propositions 4.3.1 and 4.3.2 producing from a Zin_γ-algebra respectively a Com_γ-algebra and a γ-polydendriform algebra are functors from the category of Zin_γ-algebras respectively to the category of Com_γ-algebras and the category of γ-polydendriform algebras. These functors respectively translate into symmetric operad morphisms from Com_γ to Zin_γ and from Dendr_γ to Zin_γ. These morphisms are generalizations of known morphisms between Com, Dendr, and Zin of (4.3.1) (see [Lod01, Lod06, Zin12]).

A complete study of the operads Com_γ, Lie_γ, Zin_γ, and Leib_γ, and suitable definitions for all the morphisms intervening in (4.3.2) is worth to interest for future works.

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Laboratoire d'Informatique Gaspard-Monge, Université Paris-Est Marne-la-Vallée, 5 boulevard Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, France

E-mail address: `samuele.giraudo@u-pem.fr`